W05 - Examples

Function on a random variable

Expectation of function on RV given by chart

Suppose that $g : \mathbb{R} \to \mathbb{R}$ in such a way that $g : 1 \mapsto 4$ and $g : 2 \mapsto 1$ and $g : 3 \mapsto 87$ and *no other values* are mapped to $4, 1, 87$.

$X:$	**1**	**2**	**3**
$P_X(k):$	$1/7$	$2/7$	$4/7$
$Y:$	4	1	87

Then:

$$E[X] = 1 \cdot \frac{1}{7} + 2 \cdot \frac{2}{7} + 3 \cdot \frac{4}{7} \quad \ggg \quad \frac{17}{7}$$

And:

$$E[Y] = 4 \cdot \frac{1}{7} + 1 \cdot \frac{2}{7} + 87 \cdot \frac{4}{7} \quad \ggg \quad \frac{354}{7}$$

Therefore:

$$E[5X + 2Y + 3] \quad \ggg \quad 5 \cdot \frac{17}{7} + 2 \cdot \frac{354}{7} + 3 \quad \ggg \quad \frac{814}{7}$$

Variance of uniform random variable

The uniform random variable X on $[a, b]$ has distribution given by $P\big[c \leq X \leq d\big] = \dfrac{d-c}{b-a}$ when $a \leq c \leq d \leq b$.

(a) Find $\text{Var}[X]$ using the shorter formula.

(b) Find $\text{Var}[3X]$ using "squaring the scale factor."

(c) Find $\text{Var}[3X]$ directly.

Solution
(a)

(1) Compute density.

The density for X is:

$$f_X(x) \;=\; \begin{cases} \dfrac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

(2) Compute $E[X]$ and $E[X^2]$ directly using integral formulas.

Compute $E[X]$:

$$E[X] \;=\; \int_a^b \frac{x}{b-a}\,dx \quad \gg\!\gg \quad \frac{b+a}{2}$$

Now compute $E[X^2]$:

$$E[X^2] = \int_a^b \frac{x^2}{b-a}\,dx \quad \gg\!\gg \quad \frac{1}{3}(b^2+ba+a^2)$$

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(3) Find variance using short formula.

Plug in:

$$\mathrm{Var}[X] \;=\; E[X^2] - E[X]^2$$

$$\gg\!\gg \quad \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2$$

$$\gg\!\gg \quad \frac{(b-a)^2}{12}$$

(b)

(1) "Squaring the scale factor" formula:

$$\mathrm{Var}[aX+b] \;=\; a^2\mathrm{Var}[X]$$

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(2) Plugging in:

$$\mathrm{Var}[3X] \quad \gg\!\gg \quad 9\mathrm{Var}[X] \quad \gg\!\gg \quad \frac{9}{12}(b-a)^2$$

(c)

(1) Density.

The variable $3X$ will have $1/3$ the density spread over the interval $[3a, 3b]$.

Density is then:

$$f_{3X}(x) \;=\; \begin{cases} \dfrac{1}{3b-3a} & \text{on } [3a,3b] \\ 0 & \text{otherwise} \end{cases}$$

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(2) Plug into prior variance formula.

Use $a \rightsquigarrow 3a$ and $b \rightsquigarrow 3b$.

Get variance:

$$\mathrm{Var}[3X] \;=\; \frac{(3b-3a)^2}{12}$$

Simplify:

$$\gg\!\gg \quad \frac{(3(b-a))^2}{12} \quad \gg\!\gg \quad \frac{9}{12}(b-a)^2$$

PDF of derived from CDF

Suppose that $F_X(x) = \dfrac{1}{1+e^{-x}}$.

(a) Find the PDF of X. (b) Find the PDF of e^X.

Solution

(a)

Formula:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \Longrightarrow \quad f_X(x) = \frac{d}{dx}F_X(x)$$

Plug in:

$$f_X(x) = \frac{d}{dx}\left(1+e^{-x}\right)^{-1} \quad \gg\!\gg \quad -(1+e^{-x})^{-2}\cdot(-e^{-x})$$

$$\gg\!\gg \quad \frac{e^{-x}}{\left(1+e^{-x}\right)^2}$$

(b)

By definition:

$$F_{e^X}(x) = P[e^X \leq x]$$

Since e^X is increasing, we know:

$$e^X \leq a \quad \Longleftrightarrow \quad X \leq \ln a$$

Therefore:

$$F_{e^X}(x) = F_X(\ln x)$$

$$\gg\!\gg \quad \frac{1}{1+e^{-\ln x}} \quad \gg\!\gg \quad \frac{1}{1+x^{-1}}$$

Then using differentiation:

$$f_{e^X}(x) = \frac{d}{dx}\left(\frac{1}{1+x^{-1}}\right)$$

$$\gg\!\gg \quad -(1+x^{-1})^{-2}\cdot(-x^{-2}) \quad \gg\!\gg \quad \frac{1}{(x+1)^2}$$

Probabilities via CDF

Suppose the CDF of X is given by $F_X(x) = \dfrac{1}{1+e^{-x}}$. Compute:

(a) $P[X \leq 1]$ (b) $P[X < 1]$ (c) $P[-0.5 \leq X \leq 0.2]$ (d) $P[-2 \leq X]$

Solution

Continuous wait time

Earthquake wait time

Suppose the San Andreas fault produces major earthquakes modeled by a Poisson process, with an average of 1 major earthquake every 100 years.

(a) What is the probability that there will *not* be a major earthquake in the next 20 years?

(b) What is the probability that *three* earthquakes will strike within the next 20 years?

Solution

(a)

Since the average wait time is 100 years, we set $\lambda = 0.01$ earthquakes per year. Set $X \sim \text{Exp}(0.01)$ and compute:

$$P[X > 20] = e^{-\lambda\cdot 20} \quad \gg\!\gg \quad e^{-0.01\cdot 20} \quad \gg\!\gg \quad \approx 0.82$$

(b)

The same Poisson process has the same $\lambda = 0.01$ earthquakes per year. Set $X \sim \text{Erlang}(3, 0.01)$, so:

$$f_X(t) = \frac{\lambda^\ell}{(\ell-1)!}t^{\ell-1}e^{-\lambda t}$$

$$\gg\!\gg \quad \frac{(0.01)^3}{(3-1)!}t^{3-1}e^{-0.01\cdot t} \quad \gg\!\gg \quad \frac{10^{-6}}{2}t^2e^{-0.01\cdot t}$$

and compute:

$$P[X \leq 20] = \int_0^{20} f_X(x)\,dx$$

$$\gg\!\gg \quad \int_0^{20} \frac{10^{-6}}{2}t^2e^{-0.01\cdot t}\,dt \quad \gg\!\gg \quad \approx 0.00115$$