Poisson Process

related Like Geo & Pascal

- Poission vars
- Exponential vars
- Erlang vars

Poisson Process: continuous limit of binomial:
cons avg. rate of "arrivals" on a continuum.
windows are independent

Binomial: n repetitions of trials = "window"
Poisson: window size in a continuum

PMF: $X \sim Pois(\lambda)$, X counts # arrivals in a window
avg. rate that window is λ.

$$P_X(k) = \begin{cases} e^{-\lambda} \frac{\lambda^k}{k!} & k = 0, 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

Example: Clients arriving at Post Office.
Avg. rate = 5 clients / hr.

(a) Prob. nobody in first 10 mins of opening.
(b) Prob. of 5 clients in first hour. (exactly 5)
(c) Prob. of 9 clients in first two hours. (exactly 9)

Solution: (a) window size = $1/6$ hr, $\lambda = 5/6$.

$X \sim Pois(\frac{5}{6})$, $P_X(0) = e^{-5/6} \cdot \frac{(5/6)^0}{0!} = e^{-5/6} \approx \boxed{0.435}$

(b) $\lambda = 5$, $P_X(5) = e^{-5} \frac{5^5}{5!} \approx \boxed{0.175}$

$X \sim Poi(5)$

$X \sim \mathrm{Pois}(10)$

(c) $\lambda = 10 \frac{\text{clients}}{\text{window of 2hrs}} \rightsquigarrow P_X(9) = e^{-10}\frac{10^9}{9!} \approx \boxed{0.125}$

Compare to Binomial:

Claim: $\text{Binomial}\left(n, p = \frac{\lambda}{n}\right) \longrightarrow \mathrm{Pois}(\lambda) \quad n \to \infty$

$X_n \sim \mathrm{Bin}\left(n, p = \frac{\lambda}{n}\right), \quad Y \sim \mathrm{Pois}(\lambda)$

$$P_{X_n}(k) \longrightarrow P_Y(k) \quad n \to \infty$$

E.g.: Say $\lambda = 3$, look at $P_{X_n}(1)$ as $n \to \infty$:

From Binomial PMF: $P_{X_n}(k) = \binom{n}{k} p^k q^{n-k}$

$$\rightsquigarrow P_{X_n}(1) = \binom{n}{1}\left(\frac{3}{n}\right)^1\left(1-\frac{3}{n}\right)^{n-1}$$

$\left(1+\frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x$

$$= n \cdot \frac{3}{n}\left(1-\frac{3}{n}\right)^{n-1} = 3\left(1-\frac{3}{n}\right)^{n-1} \longrightarrow 3e^{-3} \text{ as } n \to \infty$$

$X \sim Exp(\lambda)$, X measures wait time until first arrival in a Poisson process.

PDF:

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Recall: $P[a \leq X \leq b] = \int_a^b f_X \, dt$

- analog of geometric for continuous case.
- Quickly calculate "tail probability" $P[X > t] = e^{-\lambda t}$
  i.e. "exponential decay"

$X \sim Erlang(\ell, \lambda)$ when X measures wait time until ℓ^{th} arrival in Poisson process.

PDF:

$$f_X(t) = \begin{cases} \frac{\lambda^{\ell}}{(\ell-1)!} t^{\ell-1} e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Example: San Andreas fault: avg. $1 \frac{\text{quake}}{100 \text{ yrs}}$

(a) Prob. no quake next 20 years?

(b) Prob. at least three quakes next 20 years?

Solution: (a) set $\lambda = \frac{1}{100} = .01 \frac{\text{quakes}}{\text{year}}$ $X \sim Exp(.01)$

$$P[X > 20] = \int_{20}^{\infty} (.01) e^{-0.01t} \, dt = e^{-(.01) \cdot 20} \approx \boxed{0.82}$$

(b) Again $\lambda = .01$. Now:

3rd quake

$X \sim \text{Erlang}(3, .01)$

$$f_X(t) = \frac{(0.01)^3}{(3-1)!} t^{3-1} e^{-.01 \cdot t} = \frac{10^{-6}}{2} t^2 e^{-.01t}$$

$$P[X \leq 20] = \int_0^{20} \frac{10^{-6}}{2} t^2 e^{-.01t} \, dt \approx \boxed{1.15 \times 10^{-3}}$$

Derived Random Variables

Idea: $Y = g(X)$ $\quad g: \mathbb{R} \to \mathbb{R}$

OR $Z = g(X, Y)$ $\quad g: \mathbb{R}^2 \to \mathbb{R}$

technical meaning: $\underset{\text{outcome } s \in S}{Y(s)} := (g \circ X)(s)$

possible values of RV X = x such that $P_X(x) \neq 0$ or $f_X(x) \neq 0$

Some facts about expectation & variance:

- $E[g(X)] = \sum_{x = \text{pos. values}} g(x) P_X(x) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

$Y = g(X)$, $E[Y] = \sum y P_Y(y) = \sum y P_{g(X)}(y)$
$= \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} y f_{g(X)}(y) dy$

say $Y = aX + b$
or $Z = X + Y$

- $\left.\begin{aligned} E[aX+b] &= aE[X] + b \\ E[X+Y] &= E[X] + E[Y] \end{aligned}\right\}$ "linearity of expectation"

Example: $S = Bin(n, p)$, what is $E[S]$?

Solution: $S = X_1 + \ldots + X_n \quad X_i \sim Ber(p)$

$E[X_i] = p$, $E[S] = E[X_1] + \ldots + E[X_n]$
$= p + \ldots + p = \boxed{np}$

⚠ $E[XY] \neq E[X]E[Y]$ (except... indep.)

- $Var[aX + b] = a^2 Var[X]$

Pf: see notes... (use definition)

"Var squares the scale factor"

"Var kills the shift"

Example:

X	1	2	3
P_X	$1/7$	$2/7$	$4/7$
Y	4	1	87

Say $Y = g(X)$, $g: 1 \mapsto 4,\ 2 \mapsto 1,\ 3 \mapsto 87$

$$E[X] = 1 \cdot \frac{1}{7} + 2 \cdot \frac{2}{7} + 3 \cdot \frac{4}{7} = \frac{17}{7}$$

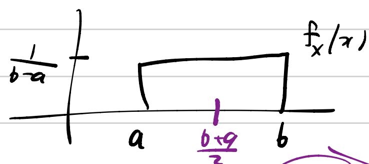
$$E[Y] = 4 \cdot \frac{1}{7} + 1 \cdot \frac{2}{7} + 87 \cdot \frac{4}{7} = \frac{354}{7}$$

Now... $E[5X + 2Y + 3] = 5E[X] + 2E[Y] + 3$

$$= 5 \cdot \frac{17}{7} + 2 \cdot \frac{354}{7} + 3 = \boxed{\frac{814}{7}}$$

Example: (See notes for details.)

$X \sim Unif[a, b] \rightsquigarrow$



$$E[X] = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b+a}{2} = \text{midpoint!}$$

$Var[X] = ?$

$$= E[X^2] - E[X]^2 \qquad \text{use } Y = g(X) = X^2$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

$$\rightsquigarrow Var[X] = \frac{1}{3}\left(b^2 + ba + a^2\right) - \left(\frac{b+a}{2}\right)^2 = \boxed{\frac{(b-a)^2}{12}}$$

Lastly: $\quad f_X \underset{?}{\overset{?}{\rightleftarrows}} f_Y \qquad Y = g(X)$

Really: $\quad f_X \longrightarrow f_{g(X)}$

⚠ NOT $g(f_X)$!

Three step process:

$$P[X \le x]$$

1. $f_X \rightsquigarrow F_X$ using $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$

2. $P[g(X) \le x] = P[X \le g^{-1}(x)]$ when g is increasing.... (e.g. $g(x) = x^2 \quad x \ge 0$)

3. $f_Y = \frac{d}{dx} F_Y(x) = \frac{d}{dx} F_X(g^{-1}(x)) \qquad f_X(x) = \frac{d}{dx} F_X(x)$

Example: Say $F_X(x) = \frac{1}{1+e^{-x}}$

(a) Find f_X $\qquad$ (b) Find f_Y: $Y = g(X) = e^X$ ✓ $g(X)$ is increasing!

Solution: (a) $f_X = \frac{d}{dx} F_X \rightsquigarrow \frac{d}{dx}(1+e^{-x})^{-1} = -(1+e^{-x})^{-2}(-e^{-x})$

$$= \boxed{\frac{e^{-x}}{(1+e^{-x})^2}}$$

(b) $F_Y(x) = P[e^X \le x] = P[X \le \ln x]$

$$= F_X(\ln x) = \frac{1}{1+e^{-\ln x}} = \frac{1}{1+x^{-1}}$$

$$= (1+x^{-1})^{-1}$$

$\rightsquigarrow f_Y = \frac{d}{dx} F_Y = \frac{d}{dx}\left((1+x^{-1})^{-1}\right) = -(1+x^{-1})^{-2}(-x^{-2})$

$$= \frac{1}{x^2(1+x^{-1})^2} = \boxed{\frac{1}{(x+1)^2}}$$