W06 Notes

Continuous families: summary

01 Theory

Uniform: $X \sim \text{Unif}([a,b])$

- All times $a \leq t \leq b$ equally likely.
- $f_X(t) = \dfrac{1}{b-a}$
- $E[X] = \dfrac{a+b}{2}$
- $\text{Var}[X] = \dfrac{1}{12}(b-a)^2$

Exponential: $X \sim \text{Exp}(\lambda)$

- Measures wait time until first arrival.
- $f_X(t) = \lambda e^{-\lambda t}$
- $E[X] = \dfrac{1}{\lambda}$
- $\text{Var}[X] = \dfrac{1}{\lambda^2}$

Erlang: $X \sim \text{Erlang}(\ell, \lambda)$

- Measures wait time until ℓ^{th} arrival.
- $f_X(t) = \dfrac{\lambda^\ell}{(\ell-1)!} t^{\ell-1} e^{-\lambda t}$
- $E[X] = \dfrac{\ell}{\lambda}$
- $\text{Var}[X] = \dfrac{\ell}{\lambda^2}$

Normal: $X \sim \mathcal{N}(\mu, \sigma^2)$

- Limiting distribution of large sums.
- $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$
- $E[X] = \mu$
- $\text{Var}[X] = \sigma^2$

Normal distribution

02 Theory

Normal distribution

A variable X has a **normal distribution**, written $X \sim \mathcal{N}(\mu, \sigma^2)$ or "X is Gaussian (μ, σ)," when it has PDF given by:

$$f_X(x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

The **standard normal** is $Z \sim \mathcal{N}(0,1)$ and its PDF is **usually denoted by $\varphi(x)$**:

$$\varphi(x) \;=\; \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The standard normal CDF is **usually denoted by $\Phi(z)$**:

$$\Phi(z) \;=\; \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

- To show that $\varphi(x)$ is a valid probability density, we must show that $\int_{-\infty}^{+\infty} \varphi(x)\, dx = 1$.
  - This calculation is *not trivial*; it requires a double integral in polar coordinates!
- There is *no explicit antiderivative* of φ
  - A computer is needed for numerical calculations.
  - A *chart of approximate values* of Φ is provided for exams.

- To check that $E[Z] = 0$:
  - Observe that $x\varphi(x)$ is an *odd function*, i.e. symmetric about the y-axis.
  - One must then simply verify that the improper integral converges.
- To check that $\mathrm{Var}[Z] = 1$:
  - Since $\mu = E[Z] = 0$, we find:

$$\mathrm{Var}[Z] = E[Z^2] \quad \gg\!\gg \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2}\, dx =: I$$

- Use integration by parts to compute that $I = 1$. (Select $u = x$ and $dv = xe^{-x^2/2}\, dx$.)

General and standard normals

Assume that $Z \sim \mathcal{N}(0,1)$ and σ, μ are constants. Define $X = \sigma Z + \mu$. Then:

$$f_X \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

That is, $\sigma Z + \mu$ has the distribution type $\mathcal{N}(\mu, \sigma^2)$.

Derivation of PDF of $\sigma Z + \mu$

Suppose that $X = \sigma Z + \mu$. Then:

$$\begin{aligned} F_X(x) &= P[X \leq x] \\ &= P[\,\sigma Z + \mu \leq x\,] \\ &= P[\,Z \leq \tfrac{x-\mu}{\sigma}\,] \\ &= \Phi\left(\tfrac{x-\mu}{\sigma}\right) \end{aligned}$$

Differentiate to find f_X:

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \end{aligned}$$

From this fact we can infer that $E[X] = \mu$ and $\text{Var}[X] = \sigma^2$ whenever $X \sim \mathcal{N}(\mu, \sigma^2)$.

03 Illustration

☰ Example - Basic generalized normal calculation

Suppose $X \sim \mathcal{N}(-3, 4)$. Find $P[X \geq -1.7]$.

Solution

First write X as a linear transformation of Z:

$$X \sim 2Z - 3$$

Then:

$$X \geq -1.7 \quad \Longleftrightarrow \quad Z \geq 0.65$$

Look in a table to find that $\Phi(0.65) \approx 0.74$ and therefore:

$$\begin{aligned} P[Z \geq 0.65] \quad &\gg\gg \quad 1 - P[Z \leq 0.65] \\ \gg\gg \quad &\approx 1 - 0.74 \quad \gg\gg \quad \boxed{0.26} \end{aligned}$$

☰ Example - Gaussian: interval of $2/3$

Find the number a such that $P\big[-a \leq Z \leq +a\big] = 2/3$.

Solution

First convert the question:

$$\begin{aligned} P\big[-a \leq Z \leq +a\big] \quad &\gg\gg \quad F_Z(a) - F_Z(-a) \\ &\gg\gg \quad \Phi(a) - \Phi(-a) \\ &\gg\gg \quad 2\Phi(a) - 1 \end{aligned}$$

Solve for a so that this value is $2/3$:

$$2\Phi(a) - 1 = 2/3 \quad \gg\gg \quad \Phi(a) = 5/6 \quad \gg\gg \quad a = \Phi^{-1}(5/6)$$

Use a Φ table to conclude $a \approx 0.97$.

☰ Example - Heights of American males

Suppose that the height of an American male in inches follows the normal distribution $\mathcal{N}(71, 6.25)$.

(a) What percent of American males are over 6 feet, 2 inches tall?

(b) What percent of those over 6 feet tall are also over 6 feet, 5 inches tall?

Solution

(a)
Let H be a random variable measuring the height of American males in inches, so $H \sim \mathcal{N}(71, 2.5^2)$. Thus $H \sim 2.5Z + 71$, and:

$$\begin{aligned} P[H > 74] &\ggg 1 - P[H \leq 74] \\ &\ggg 1 - P[2.5Z + 71 \leq 74] \\ &\ggg 1 - P[Z \leq 1.20] \\ &\ggg 1 - 0.8849 \approx 11.5\% \end{aligned}$$

(b)
We seek $P[H > 77 \mid H > 72]$ as the answer. Compute as follows:

$$\begin{aligned} P[H > 77 \mid H > 72] &= \frac{P[H > 77]}{P[H > 72]} \\ &\ggg \frac{P[2.5Z + 71 > 77]}{P[2.5Z + 71 > 72]} \\ &\ggg \frac{1 - P[Z \leq 2.4]}{1 - P[Z \leq 0.4]} = \frac{1 - 0.9918}{1 - 0.6554} \approx 2.38\% \end{aligned}$$

☰ Example - Variance of normal from CDF table

Suppose $X \sim \mathcal{N}(5, \sigma^2)$, and suppose you know $P[X > 9] = 0.2$.

Find the approximate value of σ using a Φ table.

Solution

$$X \sim \mathcal{N}(5, \sigma^2) \implies X \sim \sigma Z + 5$$

So $1 - P[X \leq 9] = 0.2$ and thus $P[\sigma Z + 5 \leq 9] = 0.8$. Then:

$$P[\sigma Z + 5 \leq 9] = P[Z \leq 4/\sigma]$$

so $P[Z \leq 4/\sigma] = 0.8$.

Looking in the chart of Φ for the nearest inverse of 0.8, we obtain $4/\sigma = 0.842$, hence $\sigma = 4.75$.