Normal Distribution

$X \sim \mathcal{N}(\mu, \sigma^2)$ -OR- "X is Gaussian (μ, σ)"

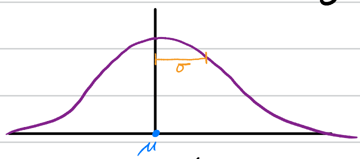
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \rightarrow -\tfrac{1}{2}\left(\tfrac{x-\mu}{\sigma}\right)^2$$

general μ, σ

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

"standard" $\mu = 0$, $\sigma = 1$

$Z \sim \mathcal{N}(0,1)$, $f_Z(z) =$ "$\varphi(z)$"



$$F_Z(z) = \text{"}\Phi(z)\text{"} = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

Standard normal CDF

FACTS:

- $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1 = \Phi(\infty)$
- $E[Z] = \mu = 0$ i.e. $\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 0$
- $Var[Z] = \sigma^2 = 1$ use IBP $u = x$, $v' = x e^{-x^2/2}$

- Suppose $Z \sim \mathcal{N}(0,1)$, $\sigma > 0$, μ are constants.

Suppose $X = \sigma Z + \mu$. Then $X \sim \mathcal{N}(\mu, \sigma^2)$

($\rightsquigarrow$ $X \sim \mathcal{N}(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma}$
Then $Z = \mathcal{N}(0,1)$.)

Pf: $F_X(x) = P[X \le x] = P[\sigma Z + \mu \le x] = P[Z \le \frac{x-\mu}{\sigma}] = \Phi\left(\frac{x-\mu}{\sigma}\right)$

$$f_X = \frac{d}{dx} F_X = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \square$$

$\sigma = 2$

Example: Say $X \sim \mathcal{N}(-3, 4)$. Find $P[X \geq -1.7]$.

Solution: Write $X = 2Z - 3$ for $Z \sim \mathcal{N}(0,1)$.

$$P[X \geq -1.7] = P[2Z - 3 \geq -1.7] \qquad \text{(sub } X = 2Z-3\text{)}$$

$$= P\left[Z \geq \frac{-1.7+3}{2}\right] \qquad \text{(solve for } Z\text{)}$$

$$= P[Z \geq 0.65]$$

can add equals b/c $P[Z = 0.65] = 0$

$$= 1 - P[Z \leq 0.65] = 1 - \Phi(0.65)$$

$$\approx 1 - 0.74 \approx \boxed{0.26}$$

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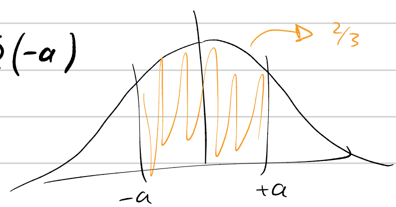
Example: Say $P[-a \leq Z \leq +a] = 2/3$. Find a.

Solution: $P[-a \leq Z \leq +a] = \Phi(a) - \Phi(-a)$

(in general $P[a \leq X \leq b] = F_X(b) - F_X(a)$)



know: $\Phi(-a) = 1 - \Phi(+a)$

$$\rightsquigarrow \Phi(a) - (1 - \Phi(a)) = 2/3 \quad \rightsquigarrow 2\Phi(a) - 1 = 2/3$$

$$\rightsquigarrow \Phi(a) = 5/6 \quad \rightsquigarrow a = \Phi^{-1}(5/6) \approx \boxed{0.97}$$

Example: Suppose American basketball players heights follow $\mathcal{N}(71, 6.25) \sim H$

(a) What % over $6'2''$? (b) Of those $\geq 6'$, what % $\geq 6'5''$?

"$X = \sigma Z + \mu$"

Solution: $\mu = 71$, $\sigma = 2.5$ b/c $\sigma^2 = 6.25$; $\underline{H = 2.5 Z + 71}$

(a) $6'2'' = 74''$, want $P[H > 74]$.

$$\rightsquigarrow P[2.5Z + 71 > 74]$$
$$\rightsquigarrow P\left[Z > \frac{74-71}{2.5}\right] \longrightarrow P[Z > 1.20] \rightsquigarrow 1 - P[Z \leq 1.20]$$

So $P[H > 74] = \boxed{1 - \Phi(1.20)}$ $\quad \Phi(z) = P[Z \leq z] = F_Z(z)$

$$\approx 1 - 0.8849$$
$$\approx \boxed{11.5\%}$$

(b) $$P[H \geq 77 \mid H \geq 72] = \frac{P[H \geq 77 \text{ AND } H \geq 72]}{P[H \geq 72]}$$

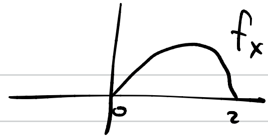
$$= \frac{P[H \geq 77]}{P[H \geq 72]} \rightsquigarrow \frac{P[2.5Z + 71 \geq 77]}{P[2.5Z + 71 \geq 72]}$$

$$\rightsquigarrow \frac{P[Z \geq 2.4]}{P[Z \geq 0.4]} \rightsquigarrow \frac{1 - P[Z < 2.4]}{1 - P[Z < 0.4]} \overset{\text{(continuous)}}{=} \frac{1 - P[Z \leq 2.4]}{1 - P[Z \leq 0.4]}$$

$$\rightsquigarrow \frac{1 - \Phi(2.4)}{1 - \Phi(0.4)} \approx \boxed{2.38\%}$$

Review

- OS-B Q6(b) ✓
- What is an RV?
- What is a distribution?
- PMF/PDF & CDF
- Unif[0,1] etc.
- Exp(λ) & Geo(p)
- $f_X \rightsquigarrow f_Y$ ✓



OS-B Q6(b) (also $f_X \rightsquigarrow f_Y$)

Have $f_X = \begin{cases} \frac{3}{4}x(2-x) & 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$ Want f_{X^2}.

Solution: $Y = g(X) = X^2$

1. Get F_X:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

$$= \int_0^x \frac{3}{4}u(2-u)\,du \qquad 0 \le x \le 2$$

$$\rightsquigarrow \frac{3}{4}u^2 - \frac{1}{4}u^3 \Big|_0^x \rightsquigarrow \frac{3}{4}x^2 - \frac{1}{4}x^3 = \frac{1}{4}x^2(3-x)$$

$$F_X(x) = \frac{1}{4}x^2(3-x)$$

2. Def. of CDF & convert:

valid b/c $Y = X^2$ is increasing $X \ge 0$

$$F_Y(x) = P[X^2 \le x] = P[X \le \sqrt{x}] = F_X(\sqrt{x})$$

$$\rightsquigarrow \frac{1}{4}(\sqrt{x})^2(3-\sqrt{x}) = \underline{\frac{1}{4}x(3-\sqrt{x})}$$

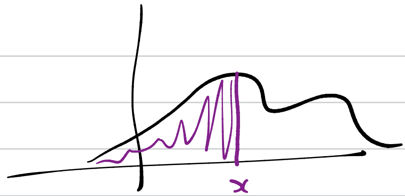
3. Get back f_Y by d/dx:

$$f_Y = \frac{d}{dx}F_Y(x) = \frac{d}{dx}\left(\frac{1}{4}x(3-\sqrt{x})\right) = \frac{3}{4} - \frac{3}{8}\sqrt{x}$$

i.e.
$$f_Y(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{4} - \frac{3}{8}\sqrt{x} & 0 \le x < 4 \\ 0 & 4 \le x \end{cases}$$

$0 \le X \le 2$
$\iff$
$0 \le X^2 \le 4$
$\iff$
$0 \le Y \le 4$

CDF



$$F_X(x) = P[X \le x]$$

$$= \int_{-\infty}^{x} f_X(u)\,du$$

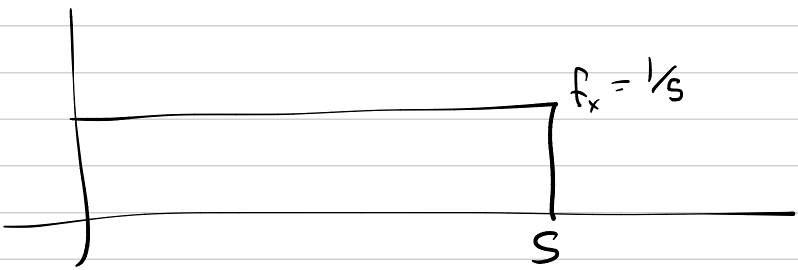
$$= \sum_{u \le x} P_X(u)$$

$$P[a \le X \le b]$$

$$\shortparallel$$

f_X and F_X FTC: $\int_a^b f_X\,du = F_X(b) - F_X(a)$

$X \sim \text{Unif}[0, s]$:



$$f_X = \begin{cases} 1/s & x \in [0,s] \\ 0 & \text{else} \end{cases}$$

so $\int_0^s 1/s\,dx = 1 =$ total probability