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Joint Distributions

- Joint PMF $P_{X,Y}(x,y) = P[X=x, Y=y]$
- Joint PDF $f_{X,Y}(x,y) =$ density at (x,y)

Say $B =$ set of (x,y) values $=$ region in X,Y chart

$$P[B] = P[(X,Y) \in B] = \sum_{(x,y)\in B} P_{X,Y}(x,y)$$

Say $\mathcal{V} =$ region in X,Y plane

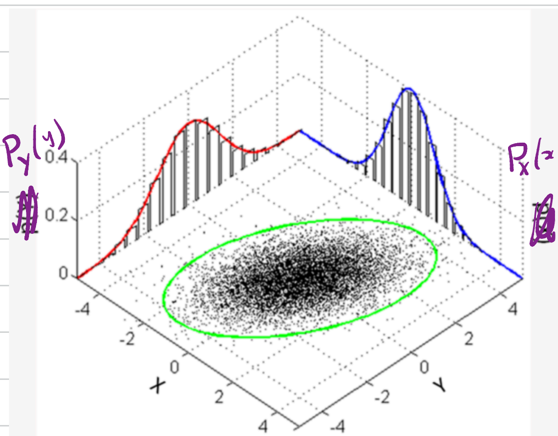
$$P[\mathcal{V}] = \iint_{\mathcal{V}} f_{X,Y}(x,y)\,dx\,dy$$

When $\mathcal{V} = \{a \le X \le b,\ c \le Y \le d\}$ have:

$$P[\mathcal{V}] = \int_c^d \int_a^b f_{X,Y}\,dx\,dy$$

Marginal distribution:

"Marginal" = PMF of X or Y considered separately.



$$P_X(x) = \sum_{\text{all } y} P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_{\text{all } x} P_{X,Y}(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

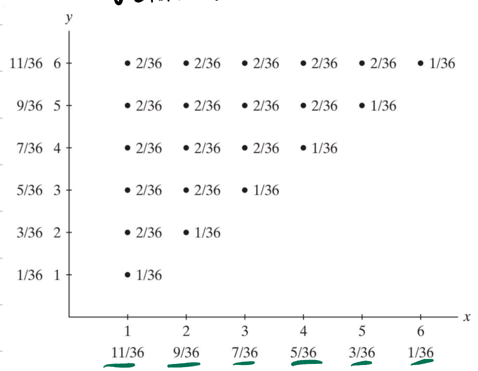
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Example: Roll 2 dice, $X = \min$ $Y = \max$.

Q: Find marginal PMFs.

A:

Joint PMF Chart:



$P_Y(6) = 11/36$
$P_Y(5) = 9/36$
$P_Y(4) = 7/36$
$P_Y(3) = 5/36$
$P_Y(2) = 3/36$
$P_Y(1) = 1/36$

Example:

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$	
$q=0$	0.06	0.18	0.24	0.12	$\rightarrow$ 0.6
$q=1$	0.04	0.12	0.16	0.08	$\rightarrow$ 0.4
	$\downarrow$ 0.1	$\downarrow$ 0.3	$\downarrow$ 0.4	$\downarrow$ 0.2	

Q:

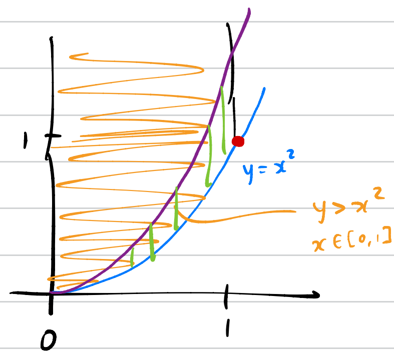
(a) $P[Q=0]$ (b) $P[Q=G]$ (c) $P[G>1]$ (d) $P[G>Q]$

A: (a) 0.6
(b) 0.18
(c) 0.6
(d) 0.78

Example: $f_{X,Y}(x,y) = \begin{cases} 2xe^{x^2-y} & y > x^2, \quad x \in [0,1] \\ 0 & \text{else} \end{cases}$

Find $f_Y(y)$ and $P[Y < 3X^2]$.

Answer: $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$



Draw region:

So $f_Y(y) = \int_{x=0}^{\sqrt{y}} 2xe^{x^2-y}\,dx$

$\rightsquigarrow e^{x^2-y}\Big|_0^{\sqrt{y}} \rightsquigarrow e^0 - e^{-y} = 1 - e^{-y}$

when $y \in [0,1]$

$f_Y(y) = \int_{x=0}^{1} 2xe^{x^2-y}\,dx \rightsquigarrow e^{x^2-y}\Big|_0^1 = e^{1-y} - e^{-y}$

$= e^{-y}(e-1)$

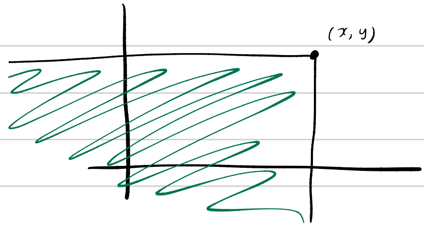
when $y > 1$

$$\boxed{f_Y = \begin{cases} 0 & y < 0 \\ 1 - e^{-y} & 0 \le y \le 1 \\ (e-1)e^{-y} & 1 < y \end{cases}}$$

$P[Y < 3X^2] = \int_{x=0}^{1} \int_{y=x^2}^{3x^2} 2xe^{x^2-y}\,dy\,dx \rightarrow \boxed{\frac{1}{2}(1+e^{-2})}$

Joint CDF

$$F_{X,Y}(x,y) = P[X \le x,\ Y \le y]$$



$$= \sum_{\substack{k \le x \\ \ell \le y}} P_{X,Y}(k,\ell)$$

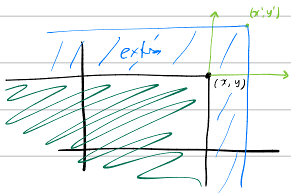
$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, du\, dv$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Example: (a) Why is $F_{X,Y}(x,y) \le F_{X,Y}(x',y')$?

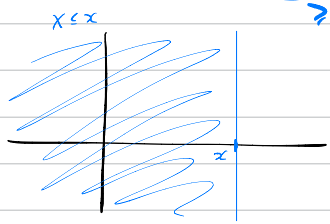
when $x < x'$, $y < y'$

ANS:



$$F_{X,Y}(x',y') = F_{X,Y}(x,y) + \text{extra}$$

$\text{extra} \geq 0$



$F_X(x)$
$= P[X \le x]$

(b) $F_X(x) = F_{X,Y}(x, \infty)$

$F_Y(y) = F_{X,Y}(\infty, y)$

(c) $F_{X,Y}(x, -\infty) = 0 = F_{X,Y}(-\infty, y)$

Main application: find $f_{R,\Theta}$ etc.

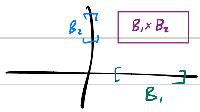
Independent RVs

Recall for events A, B: $P[A|B] = P[A] \Leftrightarrow \boxed{P[AB] = P[A]P[B]}$

Definition: X, Y independent when:

$$P[X \in B_1, Y \in B_2] = P[X \in B_1] \cdot P[Y \in B_2]$$

for all $B_1, B_2 \subset \mathbb{R}$ intervals



Too hard to check all B_1, B_2!

Short version: indep. when any of these function equalities holds:

- $P_{X,Y} = P_X \cdot P_Y$
- $f_{X,Y} = f_X \cdot f_Y$
- $F_{X,Y} = F_X \cdot F_Y$

i.e. $P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$

⚠ watch out piecewise fcns!

Example: • A man & woman plan to meet btwn 12, 1 pm.
• Uniform dist. over 12:00–1:00 each.
• Independent of each other.

Find prob. the first to arrive waits longer than 15 mins for the second to arrive.

Solution: Let X = arrival of man, $X \in (0,60)$
Y = arrival of woman, $Y \in (0,60)$

$$ANS = P[\text{"1st waits > 15 mins"}] = P[\text{"time btwn } X, Y > 15\text{"}]$$

$$= P[|X-Y| > 15]$$

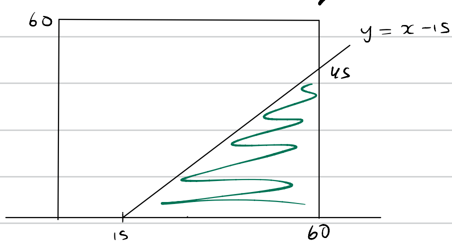
$$= P[X-Y > 15 \text{ OR } -(X-Y) > 15]$$

$$= P[X-Y > 15 \text{ OR } X-Y < -15]$$

$$= P[X-Y > 15] + P[X-Y < -15]$$

$$= 2P[X-Y > 15] \quad (\text{by symmetry } X \leftrightarrow Y)$$

$$= 2P[Y < X - 15]$$

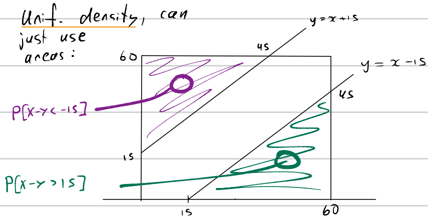


$$= 2\int_{y=0}^{45}\int_{y+15}^{60}\left(\frac{1}{60}\right)^2 dx\,dy$$

$$f_{X,Y} = f_X \cdot f_Y = \left(\frac{1}{60}\right)\cdot\left(\frac{1}{60}\right) \quad x, y \in (0,60)$$

$$\leadsto \boxed{\frac{9}{16}}$$

Unif. density, can just use areas:



$$\text{area} = \tfrac{1}{2}(45)(45)$$

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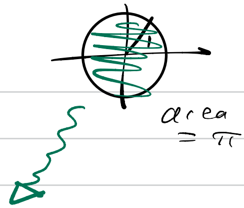
$$P[|X-Y|>15] = \frac{\text{area}+\text{area}}{60 \cdot 60}$$

$$= \frac{45^2}{60^2} = \left(\frac{3}{4}\right)^2 = \boxed{\frac{9}{16}}$$

Example: point $p = (x, y)$ Unif(unit disk) :



(a) are X, Y indep?

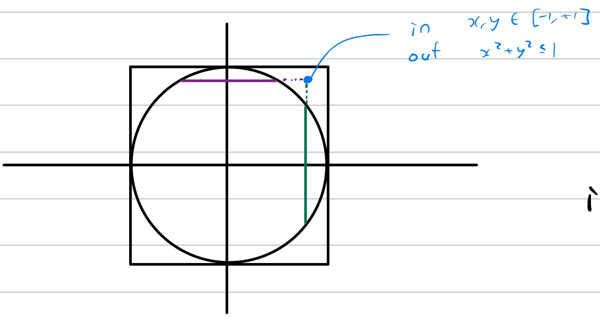
(b) are R, Θ indep?

Solution: (a) $f_{X,Y} = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \le 1 \\ 0 & \text{else} \end{cases}$

$$f_X = \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2} \qquad x \in [-1, +1]$$

$$f_Y = \frac{2}{\pi}\sqrt{1-y^2} \qquad y \in [-1, +1]$$

$$\underset{\text{on } x,y \in [-1,1]}{f_X \cdot f_Y = \frac{4}{\pi^2}\sqrt{(1-x^2)(1-y^2)}} \neq \underset{x^2+y^2 \le 1}{\frac{1}{\pi} = f_{X,Y}}$$



not independendent

(b) $f_{R,\Theta} \overset{?}{\underset{?}{=}} f_R \cdot f_\Theta$

Method :