W08 - Examples

Functions on two random variables

PMF of XY squared from chart

Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = XY^2$.

(a) Find the PMF $P_W(w)$.

(b) Find the expectation $E[W]$.

Max and Min from joint PDF

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} \frac{3}{2}(x^2+y^2) & x,\, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find the PDFs:

(a) $W = \text{Max}(X,Y)$

(b) $W = \text{Min}(X,Y)$

Solution

(a)

(1) Compute CDF of W:

Convert to event form:

$$F_W(w) \quad = \quad P\big[\text{Max}(X,Y) \leq w\big]$$

$$\gg\gg \quad P\big[X \leq w \text{ and } Y \leq w\big]$$

Integrate PDF over the region, assuming $w \in [0,1]$:

$$\int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x,y)\, dx\, dy$$

$$\gg\gg \quad \int_0^w \int_0^w \tfrac{3}{2}(x^2+y^2)\, dx\, dy \quad \gg\gg \quad w^4$$

(2) Differentiate to find $f_W(w)$:

$f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b)

(1) Compute CDF of W:

Convert to event form:

$$F_W(w) \;=\; P\big[\text{Min}(X,Y) \leq w\big]$$

$$\gg\gg \quad 1 - P\big[\text{Min}(X,Y) > w\big]$$

$$\gg\gg \quad 1 - P[X > w \text{ and } Y > w]$$

Integrate PDF over the region:

$$P[X > w \text{ and } Y > w] \quad \gg\gg \quad \int_w^1 \int_w^1 \tfrac{3}{2}(x^2 + y^2)\, dx\, dy$$

$$\gg\gg \quad w^4 - w^3 - w + 1$$

Therefore:

$$F_W(w) \;=\; w + w^3 - w^4$$

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(2) Differentiate to find $f_W(w)$:

$f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \quad = \quad \begin{cases} 1 + 3w^2 - 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

PDF of a quotient

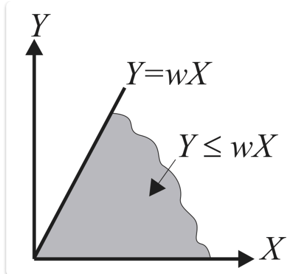
Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x + \mu y)} & x,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of $W = Y/X$.

Solution

(1) Find the CDF using logic:



Convert to event form:

$$F_W(w) \;=\; P[Y/X \leq w]$$

$$\gg\gg \quad P[Y \leq wX]$$

Integrate over this region:

$$\begin{aligned} P[Y \leq wX] \quad &= \quad \int_0^\infty \int_0^{wx} f_{X,Y}(x,y)\,dy\,dx \\ \gg\gg \quad & \quad \int_0^\infty \lambda e^{-\lambda x} \int_0^{wx} \mu e^{-\mu y}\,dy\,dx \\ \gg\gg \quad & \quad \int_0^\infty \lambda e^{-\lambda x} \left(-e^{-\mu w x} + 1\right) dx \\ \gg\gg \quad & \quad 1 - \frac{\lambda}{\lambda + \mu w} \end{aligned}$$

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(2) Differentiate to find PDF:

Compute $\frac{d}{dw} F_W(w)$:

$$f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda + \mu w)^2} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$