W09 - Examples

Sums of random variables

Sum of parabolic random variables

Suppose X is an RV with PDF given by:

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2) & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Let Y be an independent copy of X. So $f_Y = f_X$, but Y is independent of X.

Find the PDF of $X+Y$.

Solution

The graph of $f_X(w-x)$ matches the graph of $f_X(x)$ except (i) flipped in a vertical mirror, (ii) shifted by w to the left.

When $w \in [-2,0]$, the integrand is nonzero only for $x \in [-1, w+1]$:

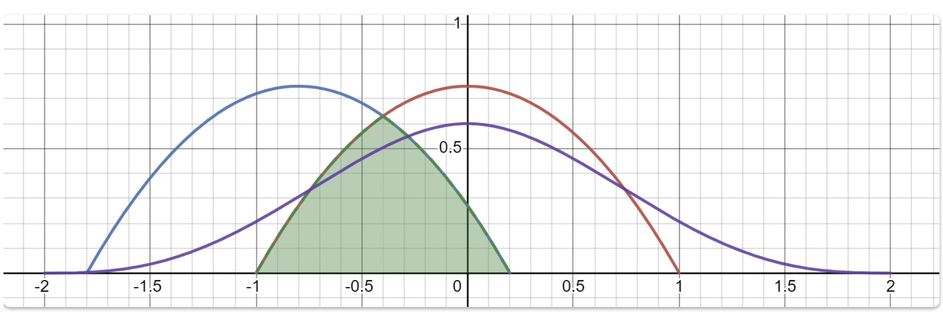
$$\begin{aligned} f_{X+Y}(w) &= \left(\frac{3}{4}\right)^2 \int_{-1}^{w+1} \left(1-(w-x)^2\right)\left(1-x^2\right) dx \\ &= \frac{9}{16}\left(\frac{w^5}{30} - \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) \end{aligned}$$

When $w \in [0,+2]$, the integrand is nonzero only for $x \in [w-1,+1]$:

$$\begin{aligned} f_{X+Y}(w) &= \left(\frac{3}{4}\right)^2 \int_{w-1}^{+1} \left(1-(w-x)^2\right)\left(1-x^2\right) dx \\ &= \frac{9}{16}\left(-\frac{w^5}{30} + \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) \end{aligned}$$

Final result is:

$$f_{X+Y}(w) = \begin{cases} \frac{9}{16}\left(\frac{w^5}{30} - \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) & w \in [-2,0] \\ \frac{9}{16}\left(-\frac{w^5}{30} + \frac{2w^3}{3} - \frac{4w^2}{3} + \frac{16}{15}\right) & w \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$



Exp plus Exp equals Erlang

Let us verify this formula by direct calculation:

$$\text{“Exp}(\lambda) + \text{Exp}(\lambda) \;\sim\; \text{Erlang}(2, \lambda)\text{”}$$

Solution

Let $X, Y \sim \text{Exp}(\lambda)$ be independent RVs.

Therefore:

$$f_X = f_Y \quad = \quad \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now compute the convolution:

$$f_{X+Y}(w) \quad = \quad \int_{-\infty}^{+\infty} f_X(w-x) f_Y(x)\, dx$$

$$\gg\gg \quad \int_0^w \lambda^2 e^{-\lambda(w-x)} e^{-\lambda x}\, dx$$

$$\gg\gg \quad \lambda^2 \int_0^w e^{-\lambda w}\, dx \quad \gg\gg \quad \lambda^2 w e^{-\lambda w}$$

This is the Erlang PDF:

$$f_X(t) = \frac{\lambda^\ell}{(\ell-1)!} t^{\ell-1} e^{-\lambda t} \bigg|_{\ell=2}$$

Expectation for two random variables

Expectation of X squared plus Y from joint PMF chart

Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = X^2 + Y$. Find the expectation $E[W]$.

Solution

First compute the values of W for each pair (X, Y) in the chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0	3
0	1	4
1	2	5

Now take the sum, weighted by probabilities:

$$\begin{array}{c} 0 \cdot (0.2) + 3 \cdot (0.2) + 1 \cdot (0.35) \\ +4 \cdot (0.1) + 2 \cdot (0.05) + 5 \cdot (0.1) \end{array} \quad \gg\gg \quad 1.95 \;=\; E[W]$$

Expectation of Y two ways and Expectation of XY from joint density

Suppose X and Y are random variables with the following joint density:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{16}xy^2 & x,y \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $E[Y]$ using two methods.

(b) Compute $E[XY]$.

Solution

(a)

(1) Method One: via marginal PDF $f_Y(y)$:

$$f_Y(y) \quad = \quad \int_0^2 \frac{3}{16}xy^2\,dx \quad \gg\gg \quad \begin{cases} \frac{3}{8}y^2 & y \in [0,2 \\ 0 & \text{otherwise} \end{cases}$$

Then expectation:

$$E[Y] \;=\; \int_0^2 y\,f_Y(y)\,dy \quad \gg\gg \quad \int_0^2 \frac{3}{8}y^3\,dy \quad \gg\gg \quad 3/2$$

(2) Method Two: directly, via two-variable formula:

$$E[Y] \;=\; \int_0^2\int_0^2 y \cdot \frac{3}{16}xy^2\,dy\,dx \gg\gg \quad \int_0^2 \frac{3}{4}x\,dx \quad \gg\gg \quad 3/2$$

(b) Directly, via two-variable formula:

$$\begin{aligned} E[XY] \quad &= \quad \int_0^2\int_0^2 xy \cdot \frac{3}{16}xy^2\,dy\,dx \\ &\gg\gg \quad \int_0^2 \frac{3}{4}x^2\,dx \quad \gg\gg \quad 2 \end{aligned}$$

Covariance from PMF chart

Suppose the joint PMF of X and Y is given by this chart:

$Y\downarrow \;\; X\rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Find $\mathrm{Cov}[X,Y]$.

Solution

We need $E[X]$ and $E[Y]$ and $E[XY]$.

$$E[X] \;=\; 1\cdot(0.2+0.35+0.05)+2\cdot(0.2+0.1+0.1) \quad \gg\gg \quad 1.4$$

$$E[Y] \;=\; -1\cdot(0.2+0.2)+0\cdot(0.35+0.1)+1\cdot(0.05+0.1)$$

$$\gg\gg \quad -0.25$$

$$E[XY] \;=\; -1\cdot(0.2)-2\cdot(0.2)+0+1\cdot(0.05)+2\cdot(0.1) \quad \gg\gg \quad -0.35$$

Therefore:

$$\mathrm{Cov}[X,Y] \;=\; E[XY]-E[X]E[Y]$$

$$\gg\gg \quad -0.35-(1.4)(-0.25) \quad \gg\gg \quad 0$$

Variance of sum of indicators

An urn contains 3 red balls and 2 yellow balls.

Suppose 2 balls are drawn without replacement, and X counts the number of red balls drawn.

Find $\mathrm{Var}[X]$.

Solution

Let X_1 indicate (one or zero) whether the first ball is red, and X_2 indicate whether the second ball is red, so $X=X_1+X_2$.

Then X_1X_2 indicates whether both drawn balls are red; so it is Bernoulli with success probability $\frac{3}{5}\frac{2}{4}=\frac{3}{10}$. Therefore $E[X_1X_2]=\frac{3}{10}$.

We also have $E[X_1]=E[X_2]=\frac{3}{5}$.

The variance sum rule gives:

$$\begin{aligned}\mathrm{Var}[X] \quad &= \quad \mathrm{Var}[X_1]+\mathrm{Var}[X_2]+2\mathrm{Cov}[X_1,X_2] \\ &\gg\gg \quad E[X_1^2]-E[X_1]^2+E[X_2^2]-E[X_2]^2+2(E[X_1X_2]-E[X_1]E[X_2]) \\ &\gg\gg \quad \frac{3}{5}-\left(\frac{3}{5}\right)^2+\frac{3}{5}-\left(\frac{3}{5}\right)^2+2\left(\frac{3}{10}-\frac{3}{5}\cdot\frac{3}{5}\right) \quad \gg\gg \quad \frac{9}{25}\end{aligned}$$