W09 Notes

Functions on two random variables

01 Theory

PMF of (any) function of two *discrete* variables

Suppose $W = g(X,Y)$ and X, Y are discrete RVs.

The PMF of W:

$$P_W(w) \;=\; \sum_{\substack{(x,y)\text{ s.t.}\\ g(x,y)=w}} P_{X,Y}(x,y)$$

CDF of (continuous) function of two *continuous* variables

Suppose $W = g(X,Y)$ and X, Y are continuous RVs, and g is a continuous function.

The CDF of W:

$$F_W(w) \;=\; P[W \leq w] \;=\; \iint_{g(x,y)\leq w} f_{X,Y}(x,y)\,dxdy$$

If desired, one can then compute the PDF of W by differentiating this CDF:

$$f_W(w) \;=\; \frac{d}{dw} F_W(w)$$

02 Illustration

Exercise - PMF of XY^2 from chart

Suppose the joint PMF of X and Y is given by this chart:

$Y \downarrow \;\; X \rightarrow$	1	2
-1	0.2	0.2
0	0.35	0.1
1	0.05	0.1

Define $W = XY^2$.

(a) Find the PMF $P_W(w)$.

(b) Find the expectation $E[W]$.

Example - Max and Min from joint PDF

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) \;=\; \begin{cases} \frac{3}{2}(x^2+y^2) & x,\, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of (a) $W = \text{Max}(X,Y)$, and of (b) $W = \text{Min}(X,Y)$.

Solution

(a)

(1) Compute CDF of W:

Convert to event form:

$$F_W(w) \;=\; P\big[\text{Max}(X,Y) \le w\big]$$

$$\gg\!\gg \quad P\big[X \le w \text{ and } Y \le w\big]$$

Integrate PDF over the region, assuming $w \in [0,1]$:

$$\int_{-\infty}^{w}\int_{-\infty}^{w} f_{X,Y}(x,y)\,dx\,dy$$

$$\gg\!\gg \quad \int_0^w\int_0^w \tfrac{3}{2}(x^2+y^2)\,dx\,dy \quad \gg\!\gg \quad w^4$$

---

(2) Differentiate to find $f_W(w)$:

$f_W = \frac{d}{dw}F_W(w)$:

$$f_W(w) \;=\; \begin{cases} 4w^3 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b)

(1) Compute CDF of W:

Convert to event form:

$$F_W(w) \;=\; P\big[\text{Min}(X,Y) \le w\big]$$

$$\gg\!\gg \quad 1 - P\big[\text{Min}(X,Y) > w\big]$$

$$\gg\!\gg \quad 1 - P[X > w \text{ and } Y > w]$$

Integrate PDF over the region:

$$P[X > w \text{ and } Y > w] \quad \gg\!\gg \quad \int_w^1\int_w^1 \tfrac{3}{2}(x^2+y^2)\,dx\,dy$$

$$\gg\!\gg \quad w^4 - w^3 - w + 1$$

Therefore:

$$F_W(w) \;=\; -w^4 + w^3 + w$$

(2) Differentiate to find $f_W(w)$:

$f_W = \frac{d}{dw} F_W(w)$:

$$f_W(w) \; = \; \begin{cases} -4w^3 + 3w^2 + 1 & w \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Example - PDF of a sum

Suppose X is an RV with density:

$$f_X \; = \; \begin{cases} 2x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Suppose Y is uniform on $[0,1]$ and independent of X.

Find the PDF of $X + Y$. Sketch the graph of this PDF.

Solution

(1) Write the CDF of $W = X + Y$ as a double integral:

$$F_W(w) \; = \; P[X + Y \leq w] \; = \; \iint_{x+y \,\leq\, w} f_{X,Y} \, dx \, dy$$

The joint density on the unit square $x \in [0,1], \; y \in [0,1]$ is:

$$f_{X,Y} \quad \ggg \quad f_X \cdot f_Y \quad \ggg \quad 2x \cdot 1 \quad \ggg \quad 2x$$

There is positive density in the region $x + y \leq w$ only for $x \leq w$ (otherwise $y < 0$).

- When $w \in [0,1]$, there is positive density in the region (only) when $y \leq w - x$.
- When $w \in [1,2]$, there is positive density in the region whenever $y \leq 1$.

---

(2) Evaluate $F_W(w)$ for $w \in [0,1]$:

Here $x \in [0,w]$ and $w - x \leq 1$, so $y \in [0, \; w - x]$.

$$F_W(w) \; = \; \int_0^w \int_0^{w-x} 2x \, dy \, dx$$

$$\ggg \quad \int_0^w 2x(w-x) \, dx \quad \ggg \quad \left[wx^2 - \tfrac{2x^3}{3} \right]_0^w$$

$$\ggg \quad w^3 - \frac{2w^3}{3} \quad \ggg \quad \frac{w^3}{3}$$

Differentiate:

$$f_W(w) \; = \; \frac{d}{dw} \frac{w^3}{3} \quad \ggg \quad w^2$$

---

(3) Evaluate $F_W(w)$ for $w \in [1,2]$:

Now x ranges over $[0, 1]$. Split by whether the y-bound is 1 or $w - x$:

- $x \in [0,\, w-1]$: $w - x \geq 1$, so $y \in [0, 1]$
- $x \in [w-1,\, 1]$: $w - x < 1$, so $y \in [0,\, w - x]$

$$F_W(w) \;=\; \int_0^{w-1}\int_0^1 2x\,dy\,dx \;+\; \int_{w-1}^1\int_0^{w-x} 2x\,dy\,dx$$

$$\gg\gg \quad \int_0^{w-1} 2x\,dx \;+\; \int_{w-1}^1 2x(w-x)\,dx$$

$$\gg\gg \quad (w-1)^2 \;+\; \left[wx^2 - \frac{2x^3}{3}\right]_{w-1}^1$$

$$\gg\gg \quad (w-1)^2 + \left(w - \tfrac{2}{3}\right) - \left(w(w-1)^2 - \tfrac{2(w-1)^3}{3}\right)$$

$$\gg\gg \quad -\frac{1}{3}x^3 + w^2 - \frac{1}{3}$$

---

(4) Differentiate for the final PDF:

$$f_W(w) \;=\; \frac{d}{dw}\left(-\frac{1}{3}x^3 + w^2 - \frac{1}{3}\right) \quad \gg\gg \quad -w^2 + 2w$$

Therefore:

$$f_{X+Y}(w) \;=\; \begin{cases} w^2 & w \in [0,1] \\ -w^2 + 2w & w \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

Extra - Convolution

Convolution

Extra Example - PDF of a quotient

Suppose the joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) \;=\; \begin{cases} \lambda\mu e^{-(\lambda x + \mu y)} & x,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

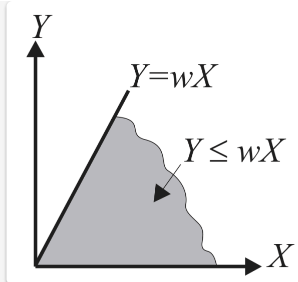
Find the PDF of $W = g(X,Y)$ for $g(X,Y) = Y/X$.

Solution

(1) Find the CDF using logic:

$$F_W(w) \;=\; P[Y/X \leq w]$$

$$\gg\gg \quad P[Y \leq wX]$$



Integrate over this region:

$$
\begin{aligned}
P[Y \leq wX] &= \int_0^\infty \int_0^{wx} f_{X,Y}(x,y)\,dy\,dx \\
&\gg\gg \int_0^\infty \lambda e^{-\lambda x} \int_0^{wx} \mu e^{-\mu y}\,dy\,dx \\
&\gg\gg \int_0^\infty \lambda e^{-\lambda x} \left(-e^{-\mu w x} + 1\right) dx \\
&\gg\gg 1 - \frac{\lambda}{\lambda + \mu w}
\end{aligned}
$$

---

(2) Differentiate to find PDF:

Compute $\frac{d}{dw} F_W(w)$:

$$
f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda + \mu w)^2} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

03 Theory

Recall that in a Poisson process:

- $X \sim \text{Exp}(\lambda)$ measures continuous wait time until *one* arrival
- $X \sim \text{Erlang}(\ell, \lambda)$ measures continuous wait time until ℓ^{th} arrival

Since the wait times between arrivals are *independent*, we expect that the *sum of exponential distributions is an Erlang distribution*. This is true!

Erlang sum rule

Specify a given Poisson process with arrival rate λ. Suppose that:

- $X \sim \text{Erlang}(r, \lambda)$ for any $r = 1, 2, 3, \ldots$
- $Y \sim \text{Erlang}(s, \lambda)$ or any $s = 1, 2, 3, \ldots$
- X and Y are independent

Then:

$$X + Y \quad \sim \quad \text{Erlang}(r+s, \lambda)$$

Exp plus Exp is Erlang

Recall that $\text{Erlang}(1, \lambda) \sim \text{Exp}(\lambda)$.

So we could say:

$$\text{“Exp}(\lambda) + \text{Exp}(\lambda) \ \sim \text{Erlang}(2, \lambda)\text{”}$$

And:

$$\text{“Exp}(\lambda) + \text{Erlang}(\ell, \lambda) \ \sim \text{Erlang}(\ell+1, \lambda)\text{”}$$

04 Illustration

Example - Exp plus Exp equals Erlang

Let us verify this formula by direct calculation:

$$\text{“Exp}(\lambda) + \text{Exp}(\lambda) \ \sim \ \text{Erlang}(2, \lambda)\text{”}$$

Solution

Let X, $Y \sim \text{Exp}(\lambda)$ be independent RVs, and let $W = X + Y$.

Therefore:

$$f_{X,Y}(x,y) \ = \ \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} \ = \ \lambda^2 e^{-\lambda(x+y)} \qquad x \geq 0, \ y \geq 0$$

---

(1) Write the CDF as a double integral over the region $x \geq 0, \ y \geq 0, \ x + y \leq w$:

For $w \geq 0$, the region is $x \in [0, w]$, $y \in [0, \ w - x]$.

$$F_W(w) \ = \ \int_0^w \int_0^{w-x} \lambda^2 e^{-\lambda(x+y)} \, dy \, dx$$

---

(2) Evaluate the inner integral:

$$\int_0^{w-x} \lambda^2 e^{-\lambda x} e^{-\lambda y} \, dy \quad \gg\gg \quad \lambda^2 e^{-\lambda x} \left[-\tfrac{1}{\lambda} e^{-\lambda y} \right]_0^{w-x}$$

$$\gg\gg \quad \lambda \, e^{-\lambda x} \big(1 - e^{-\lambda(w-x)} \big)$$

$$\gg\gg \quad \lambda \big(e^{-\lambda x} - e^{-\lambda w} \big)$$

---

(3) Evaluate the outer integral:

$$F_W(w) \ = \ \lambda \int_0^w \left(e^{-\lambda x} - e^{-\lambda w}\right) dx$$

$$\ggg \quad \lambda \left[-\tfrac{1}{\lambda} e^{-\lambda x} - x e^{-\lambda w}\right]_0^w$$

$$\ggg \quad \lambda \left[\left(-\tfrac{1}{\lambda} e^{-\lambda w} - w e^{-\lambda w}\right) - \left(-\tfrac{1}{\lambda}\right)\right]$$

$$\ggg \quad 1 - e^{-\lambda w} - \lambda w e^{-\lambda w}$$

---

(4) Differentiate for the PDF:

$$f_W(w) \ = \ \frac{d}{dw}\left(1 - e^{-\lambda w} - \lambda w e^{-\lambda w}\right)$$

$$\ggg \quad \lambda e^{-\lambda w} - \lambda e^{-\lambda w} + \lambda^2 w e^{-\lambda w}$$

$$\ggg \quad \lambda^2 w e^{-\lambda w}$$

This is the $\text{Erlang}(2, \lambda)$ density function:

$$\left.\frac{\lambda^\ell}{(\ell-1)!} t^{\ell-1} e^{-\lambda t}\right|_{\ell=2}$$

Exercise - Erlang induction step

Derive the formula:

$$\text{“Exp}(\lambda) + \text{Erlang}(\ell, \lambda) \ \sim \ \text{Erlang}(\ell+1, \lambda)\text{”}$$

Observation: By repeatedly applying the above formula, we see that:

$$\text{“}\overbrace{\text{Exp}(\lambda) + \cdots + \text{Exp}(\lambda)}^{\ell \text{ terms}} \ \sim \ \text{Erlang}(\ell, \lambda)\text{”}$$