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Functions on two RVs

Recall: $Y = g(X)$ then we have:

$$P_Y(y) = \sum_{g(x)=y} P_X(x) \qquad \Big| \qquad E[Y] = \sum_x g(x)\,P_X(x)$$

$$\sum_y y\,P_Y(y)$$

Want f_Y? $\quad F_Y(y) = P[g(X) \le y] \leftarrow$ calculate, in terms of F_X

$$\rightsquigarrow f_Y = \frac{d}{dy} F_Y$$

Now: $W = g(X, Y)$

more in W10

Discrete

$$P_W(w) = \sum_{\substack{g(x,y)=w \\ g^{-1}(w)}} P_{X,Y}(x,y) \qquad \Big| \qquad E[W] = \sum_{x,y} g(x,y)\,P_{X,Y}(x,y)$$

Continuous

$$F_W(w) = P[W \le w] = P[g(X,Y) \le w] = \iint_{g(x,y)\le w} f_{X,Y}(x,y)\,dx\,dy$$

$$\rightsquigarrow f_W = \frac{d}{dw} F_W(w)$$

Example:

Y \ X	1	2
-1	.2	.2
0	.35	.1
1	.05	.1

$W = XY^2$

(a) Find P_W

(b) Find $E[W]$

W chart:

1	2
0	0
1	2

Solution:

(a)

$$P_W(w) = \begin{cases} .45 & w = 0 \\ .25 & w = 1 \\ .3 & w = 2 \end{cases}$$

(b)

$$1\cdot .2 + 2\cdot .2 + 0\cdot .35 + 0\cdot .1 + 1\cdot .05 + 2\cdot .1 = .6 + .25 = .85$$

Example: $f_{X,Y} = \begin{cases} \frac{3}{2}(x^2+y^2) & x,y \in [0,1] \\ 0 & \text{else} \end{cases}$

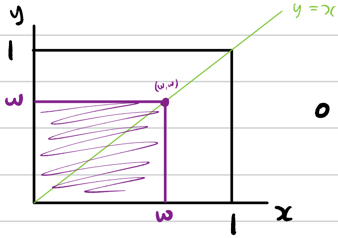
Find PDF of (a) $W = Max(X,Y)$
(b) $W = Min(X,Y)$

Solution: (a)

$\iint_{Max(x,y) \leq w} f_{X,Y}\, dx\, dy$ $\; /\!/$

- CDF first: $F_W(w) = P[Max(X,Y) \leq w]$

  - Region $Max(x,y) \leq w \iff x \leq w$ AND $y \leq w$

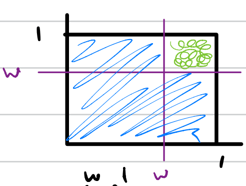


- $F_W = \int_{y=0}^{w} \int_{x=0}^{w} \frac{3}{2}(x^2+y^2)\, dx\, dy \rightsquigarrow w^4$

- Thus $\boxed{f_W = \begin{cases} 4w^3 & 0 \leq w \leq 1 \\ 0 & \text{else} \end{cases}}$

(b)

$F_W = P[Min(X,Y) \leq w] \overset{?}{=} \cancel{P[X > w \text{ AND } Y > w]}$



$\overset{\checkmark}{=} \underbrace{P[X \leq w \text{ OR } Y \leq w]}$

$\overset{\checkmark}{=} 1 - \underbrace{P[X > w \text{ AND } Y > w]}$

$\rightsquigarrow \int_0^w \int_0^1 \frac{3}{2}(x^2+y^2)\, dy\, dx + \int_w^1 \int_0^w \frac{3}{2}(x^2+y^2)\, dy\, dx$

these are equal! two methods.

$\rightsquigarrow 1 - \int_w^1 \int_w^1 \frac{3}{2}(x^2+y^2)\, dy\, dx$

$1 - (w^4 - w^3 - w + 1) \rightsquigarrow$ $\boxed{f_W = \begin{cases} -4w^3 + 3w^2 + w & 0 \leq w \leq 1 \\ 0 & \text{else} \end{cases}}$

Example: • $f_X = 2x \quad x \in [0,1] \quad$ (else 0)

• $Y \sim Unif[0,1]$ i.e. $f_Y = 1 \quad y \in [0,1]$

• Indep.

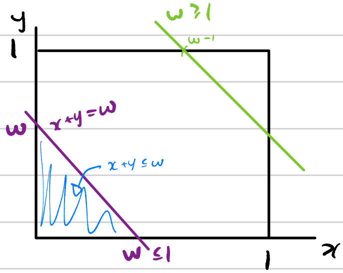
Find f_W, $W = X+Y$.

Solution:

$f_X \cdot f_Y = 2x \cdot 1 = 2x$ on $x, y \in [0,1]$

$$F_W = P[X+Y \le w] = \iint_{x+y \le w} f_{X,Y} \, dy \, dx$$

Conditions: $0 \le x, y \le 1 \qquad x+y \le w$



When $0 \le w \le 1$:

$$\int_{x=0}^{w} \int_{y=0}^{w-x} 2x \, dy \, dx \leadsto \frac{1}{3} w^3$$

When $1 \le w \le 2$:

$$\int_{0}^{w-1} \int_{0}^{1} 2x \, dy \, dx + \int_{w-1}^{1} \int_{0}^{w-x} 2x \, dy \, dx \leadsto -\frac{1}{3} w^3 + w^2 + \frac{1}{3}$$

Therefore

$$\boxed{f_W = \begin{cases} w^2 & 0 \le w \le 1 \\ -w^2 + 2w & 1 \le w \le 2 \\ 0 & \text{else} \end{cases}}$$

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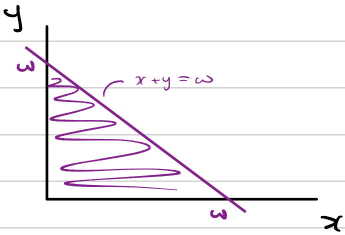
Erlang Distribution Function

$X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$, X, Y indep.

$$\rightsquigarrow X + Y \sim Erlang(2, \lambda)$$

Proof: $f_X = \lambda e^{-\lambda x}$, $f_Y = \lambda e^{-\lambda y}$ $\quad x, y \geq 0$

By indep.: $f_{X,Y} = f_X \cdot f_Y = \lambda^2 e^{-\lambda(x+y)}$ $\quad x, y \geq 0$



$$F_W = \iint_{x+y \leq \omega} \lambda^2 e^{-\lambda(x+y)} \, dy \, dx$$

$$= \int_0^{\omega} \int_0^{\omega - x} \lambda^2 e^{-\lambda(x+y)} \, dy \, dx$$

$$\rightsquigarrow \int_0^{\omega} \lambda (e^{-\lambda x} - e^{-\lambda y}) \, dx \rightsquigarrow 1 - e^{-\lambda\omega} - \lambda\omega e^{-\lambda\omega}$$

Then $f_W = \frac{d}{dw} F_W = \cancel{-(-\lambda) e^{-\lambda\omega} - \lambda e^{-\lambda\omega}} - \lambda\omega(-\lambda) e^{-\lambda\omega}$

$$\rightsquigarrow \lambda^2 \omega e^{-\lambda\omega} \qquad \omega \geq 0$$

Recall Erlang: $f_X(t) = \frac{\lambda^{\ell}}{(\ell-1)!} t^{\ell-1} e^{-\lambda t} \Big|_{\ell=2}$

$X \sim Erlang(\ell, \lambda)$ $\square$

Similarly:

"$\text{Exp}(\lambda) + \text{Erlang}(\ell, \lambda) \sim \text{Erlang}(\ell+1, \lambda)$"

<u>Proof</u>: just like above, you do it!

Therefore: Erlang Sum Rule

"$\text{Erlang}(\ell, \lambda) + \text{Erlang}(k, \lambda) \sim \text{Erlang}(\ell+k, \lambda)$"