W10 - Examples

Conditional distribution

Conditional PMF, variable event, via joint density

Suppose X and Y have joint PMF given by:

$$P_{X,Y}(k,\ell) \quad = \quad \begin{cases} \dfrac{k+\ell}{21} & k=1,2,3;\ \ell=1,2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P_{X|Y}(k|\ell)$ and $P_{Y|X}(\ell,k)$.

Solution

Marginal PMFs:

$$\begin{aligned} P_X(k) \quad &= \quad \frac{2k+3}{21}, \qquad k=1,2,3 \\ P_Y(\ell) \quad &= \quad \frac{\ell+2}{7}, \qquad \ell=1,2 \end{aligned}$$

Assuming $\ell = 1$ or 2, for each $k = 1,\ 2,\ 3$ we have:

$$P_{X|Y}(k,\ell) \quad = \quad \frac{P_{X,Y}(k,\ell)}{P_Y(\ell)} \quad \gg\gg \quad \frac{k+\ell}{3\ell+6}$$

Assuming $k = 1,\ 2,$ or 3, for each $\ell = 1, 2$ we have:

$$P_{Y|X}(\ell,k) \quad = \quad \frac{P_{Y,X}(\ell,k)}{P_X(k)} \quad \gg\gg \quad \frac{k+\ell}{2k+3}$$

Conditional expectation

Conditional PMF, fixed event, expectation

Suppose X measures the lengths of some items and has the following PMF:

$$P_X(k) \quad = \quad \begin{cases} 0.15 & k=1,2,3,4 \\ 0.1 & k=5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Let L be the event that $X \geq 5$.

(a) Find the conditional PMF of X given that L is known.

(b) Find the conditional expected value and variance of X given L.

Solution

(a)

Conditional PMF formula with $x \in L$ plugged in:

$$P_{X|L}(x) \quad = \quad \begin{cases} \dfrac{P_X(x)}{P[L]} & x = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Compute $P[L]$ by adding cases:

$$P[L] \;=\; \sum_{k=5}^{8} P_X(k) \quad \gg\gg \quad 0.4$$

Divide nonzero PMF entries by 0.1:

$$P_{X|L}(k) \quad = \quad \begin{cases} 0.25 & k = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

(b)

Find $E[\,X \mid L\,]$:

$$E[\,X \mid L\,] \;=\; \sum_{k=5}^{8} k\, P_{X|L}(k)$$

$$\gg\gg \quad 5 \cdot (0.25) + 6 \cdot (0.25) + 7 \cdot (0.25) + 8 \cdot (0.25)$$

$$\gg\gg \quad 6.5\,\text{min}$$

Find $E[\,X^2 \mid L\,]$:

$$E[\,X^2 \mid L\,] \;=\; \sum_{k=5}^{8} k^2\, P_{X|L}(k)$$

$$\gg\gg \quad 5^2 \cdot (0.25) + 6^2 \cdot (0.25) + 7^2 \cdot (0.25) + 8^2 \cdot (0.25)$$

$$\gg\gg \quad 43.5\,\text{min}^2$$

Find $\text{Var}[\,X \mid L\,]$ using “short form” with conditioning:

$$\text{Var}[\,X \mid L\,] \quad = \quad E[\,X^2 \mid L\,] - E[\,X \mid L\,]^2 \quad \gg\gg \quad 1.25\,\text{min}^2$$

Conditional expectations from joint density

Suppose X and Y are random variables with joint density given by:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{y} e^{-x/y} e^{-y} & x, y \in (0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Find $E[\,X \mid Y = y\,]$. Use this to compute $E[X]$.

Solution

(1) Derive the marginal density $f_Y(y)$:

$$f_Y(y) \quad \ggg \quad \int_0^{+\infty} \tfrac{1}{y} e^{-x/y} e^{-y}\, dx$$

$$\ggg \quad -e^{-x/y} e^{-y}\Big|_{x=0}^{\infty} \quad \ggg \quad e^{-y}$$

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(2) Use $f_Y(y)$ to compute $f_{X|Y}(x|y)$:

$$f_{X|Y}(x|y) \quad \ggg \quad \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\ggg \quad \tfrac{1}{y} e^{-x/y} e^{-y} \cdot (e^{-y})^{-1} \quad \ggg \quad \tfrac{1}{y} e^{-x/y}$$

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(3) Use $f_{X|Y}(x|y)$ to calculate expectation conditioned on the variable event:

$$E[X \mid Y = y] \quad \ggg \quad \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx$$

$$\ggg \quad \int_0^{\infty} \tfrac{x}{y} e^{-x/y}\, dx \quad \ggg \quad y$$

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(4) Apply Iterated Expectation:

Set $g(y) = y$. By Iterated Expectation, we know that $E[X] = E[g(Y)]$. Therefore:

$$E[X] = E[g(Y)] \quad = \quad \int_{-\infty}^{+\infty} g(y)\, f_Y(y)\, dy$$

$$\ggg \quad \int_0^{+\infty} y\, e^{-y}\, dy \quad \ggg \quad 1$$

Notice that $g(Y) = Y$, so $E[\,X \mid Y\,] = Y$, and Iterated Expectation says that $E[X] = E[Y]$.

Sum of random number of RVs

Let N denote the number of customers that enter a store on a given day.

Let X_i denote the amount spent by the i^{th} customer.

Assume that $E[N] = 50$ and $E[X_i] = \$8$ for each i.

What is the expected total spend of all customers in a day?

Solution

A formula for the total spend is $X = \sum_{i=1}^{N} X_i$.

By Iterated Expectation, we know $E[X] = E[\,E[\,X \mid N\,]\,]$.

Now compute $E[X \mid N]$ as a function of N:

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\begin{aligned}
E[X \mid N = n] \quad &\ggg \quad E\left[\left(\sum_{i=1}^{N} X_i\right) \mid N = n\right] \\
&\ggg \quad E\left[\left(\sum_{i=1}^{n} X_i\right) \mid N = n\right] \\
&\ggg \quad \sum_{i=1}^{n} E[X_i \mid N = n] \\
&\ggg \quad \sum_{i=1}^{n} E[X_i] \quad \ggg \quad 8n
\end{aligned}
$$

Therefore $g(n) = 8n$ and $g(N) = 8N$ and $E[\,X \mid N\,] = 8N$.

Then by Iterated Expectation, $E[X] = E[8N] = 8E[N] = \$400$.