W10 Homework B

Due date: Sunday 3/22, 11:59pm

01 ★

Correlation between overlapping coin flip sequences

Suppose a coin is flipped 30 times.

Let X count the number of heads among the first 20 flips, and Y count the heads in the last 20.

Find $\rho[X, Y]$.

Hint: Partition the flips into three groups of 10. Create *three* variables, counting heads, and express X and Y using these. What is the variance of a binomial distribution?

02 ★

Variance puzzle: indicators

Suppose A and B are events satisfying:

$$P[A] = 0.5, \qquad P[B] = 0.2, \qquad P[AB] = 0.1$$

Let X count the number of these events that occur. (So the possible values are $X = 0, 1, 2$.)

Find $\text{Var}[X]$.

Hint: Try setting $X = X_A + X_B$.

03

Covariance etc. from joint density

Suppose X and Y are random variables with the following joint density:

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}\left(x^2+y^2\right) & x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Compute:

(a) $E[X]$ (b) $E[Y]$ (c) $E[XY]$ (d) $\mathrm{Var}[X]$

(e) $\mathrm{Var}[Y]$ (f) $\mathrm{Cov}[X,Y]$ (g) $\rho[X,Y]$ (h) Are X and Y independent?

(It is worth thinking through which of these can be computed in multiple ways.)

04 - optional problem

When $\rho[X,Y]=1$

Suppose $\rho[X,Y]=1$ for two random variables X and Y.

Prove that $Y=aX+b$, where $a=\sigma_Y/\sigma_X$, and find the formula for b.

Hint: Study the derivation that $-1\leq\rho[X]\leq 1$, and think about $E[(\tilde{X}-\tilde{Y})^2]$.

(Note: A similar result and argument holds for $\rho[X,Y]=-1$.)

05

Factorizing the density

Consider two joint density functions for X and Y:

$$f_1(x,y) = 6e^{-2x}e^{-3y} \qquad x,y>0,$$
$$f_2(x,y) = 24xy \qquad x,y\in[0,1],\ x+y\in[0,1].$$

(Assume the densities are zero outside the given domain.)

Supposing f_1 is the joint density, are X and Y independent? Why or why not?

Supposing f_2 is the joint density, are X and Y independent? Why or why not?