

Expectations in Two Variables

Recall: $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x dx$ (vs. $\int_{-\infty}^{\infty} y f_y dy$)

Now: $E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{x, y} dx dy$ (continuous)

- OR - $= \sum_{k, l} g(k, l) P_{x, y}(k, l)$ (discrete)

Always: $E[ax + by + c] = aE[x] + bE[y] + c$
 (constants a, b, c)
 $\leadsto E[x + y] = E[x] + E[y]$

using $E[x + y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) f_{x, y} dx dy$
 $\leadsto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{x, y} dx dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{x, y} dx dy$
 $= E[x] + E[y]$

When Independent: $E[xy] = E[x] \cdot E[y]$

use $E[xy] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f_{x, y} dx dy$ $f_x f_y$ by independence
 $\leadsto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x f_x) (y f_y) dy dx$
 $\leadsto \int_{-\infty}^{\infty} x f_x dx \left(\int_{-\infty}^{\infty} y f_y dy \right) = E[x] E[y]$

$x \rightarrow 1 \quad 2$

| | | |
|----|-----|----|
| -1 | .2 | .2 |
| 0 | .35 | .1 |
| 1 | .05 | .1 |

Example:

Find $E[X^2 + Y]$.

Solution: Set $W = g(X, Y) = X^2 + Y$.

$x \rightarrow 1 \quad 2$

| | | |
|----|---|---|
| -1 | 0 | 3 |
| 0 | 1 | 4 |
| 1 | 2 | 5 |

$$\begin{aligned} \leadsto E[W] &= 0 \cdot (.2) + 3 \cdot (.2) \\ &+ 1 \cdot (.35) + 4 \cdot (.1) \\ &+ 2 \cdot (.05) + 5 \cdot (.1) \end{aligned}$$

$$\leadsto \boxed{1.95}$$

Example: Know: $X \sim \text{Geo}(p)$, $Y \sim \text{Bm}(n, q)$

Which do I know?

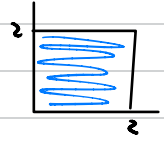
$$E[X+Y] \quad \checkmark$$

$$E[XY] \quad \times$$

$$E[X^2+Y^2] \quad \checkmark$$

$$E[(X+Y)^2] \quad \times$$

Example: $f_{x,y} = \begin{cases} \frac{3}{16} xy^2 & x,y \in [0,2] \\ 0 & \text{else} \end{cases}$



- (a) $E[Y]$
 (b) $E[XY]$

Solution:

(a) One:

$$f_y = \int_{-\infty}^{\infty} f_{x,y} dx \rightsquigarrow \int_0^2 \frac{3}{16} xy^2 dx \rightsquigarrow \begin{cases} \frac{3}{8} y^2 & y \in [0,2] \\ 0 & \text{else} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} y f_y dy \rightsquigarrow \int_0^2 y \cdot \frac{3}{8} y^2 dy \rightsquigarrow \boxed{\frac{3}{2}}$$

Two: $g(x,y) = y$

$$\rightsquigarrow E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{x,y} dx dy$$

$$\rightsquigarrow \int_0^2 \int_0^2 y \cdot \frac{3}{16} xy^2 dy dx \rightsquigarrow \int_0^2 \frac{3}{4} x dx \rightsquigarrow \boxed{\frac{3}{2}}$$

(b) $E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f_{x,y} dx dy$

$$\rightsquigarrow \int_0^2 \int_0^2 xy \frac{3}{16} xy^2 dy dx \rightsquigarrow \int_0^2 \frac{3}{4} x^2 dx \rightsquigarrow \boxed{2}$$

Covariance & Correlation

Observe: $\left. \begin{array}{l} X - \mu_x \\ Y - \mu_y \end{array} \right\}$ "centered at zero"

Define: $\text{Cov}[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$

(Recall $\text{Var}[X] = E[(X - \mu_x)^2]$)

"short formula" $= E[XY] - E[X]E[Y]$

Example:

| $x \rightarrow$ | 1 | 2 |
|-----------------|-----|----|
| -1 | .2 | .2 |
| 0 | .35 | .1 |
| 1 | .05 | .1 |

$\text{Cov}[X, Y] = ?$

Solution:

$$\begin{aligned} E[X] &= 1.4 \\ E[Y] &= -0.25 \end{aligned}$$

$$\begin{aligned} E[XY] &= -1(.2) - 2(.2) + 0(.35 + .1) \\ &\quad + 1(.05) + 2(.1) \rightsquigarrow -0.35 \end{aligned}$$

$$\rightsquigarrow E[XY] - E[X]E[Y]$$

$$\rightsquigarrow (-0.35) - (1.4)(-0.25) \rightsquigarrow \boxed{0}$$

"uncorrelated"

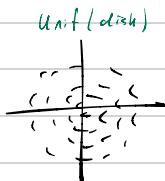
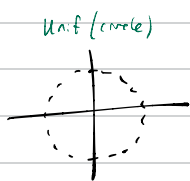
Def.: $\rho[X, Y] = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y} = \text{Cov}\left[\frac{X - \mu_X}{\sigma_X}, \frac{Y - \mu_Y}{\sigma_Y}\right]$

FACT: $-1 \leq \rho[X, Y] \leq +1$

Also: $\rho = \pm 1 \rightsquigarrow Y = aX + b$ (cf. HW)

Terms: "positively correlated" $\text{Cov}[X, Y] > 0$ or $\rho > 0$
 "negatively correlated" $\text{Cov}[X, Y] < 0$ or $\rho < 0$
 "uncorrelated" $\text{Cov}[X, Y] = 0 = \rho$

Example:



dependent
but uncorrelated

But FACT: Independence \Rightarrow Uncorrelated

- FACTS:
- $\text{Cov}[X, Y] = \text{Cov}[Y, X]$
 - $\text{Cov}[X, X] = \text{Var}[X]$

$\text{Cov}[aX + b, Y] = a \text{Cov}[X, Y]$

Redeem Variance rule:

$$\begin{aligned} \text{Var}[aX] &= \text{Cov}[aX, aX] = a \text{Cov}[X, aX] \\ &= a \text{Cov}[aX, X] = a^2 \text{Cov}[X, X] \end{aligned}$$

$\text{Cov}[X + Y, Z] = \text{Cov}[X, Z] + \text{Cov}[Y, Z]$

Intuition: Like $(x+y) \cdot z = x \cdot z + y \cdot z$

$\rho[aX + b, Y] = \text{sign}(a) \rho[X, Y]$

Theorem: "Variance Sum Rule"

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X, Y]$$

Intuition: $(x+y) \cdot (x+y) = x \cdot x + y \cdot y + 2x \cdot y$

Example:
• Urn has 3 red, 2 yellow marbles.
• Draw two without replacement.
• $X = \#$ red drawn ($X = 0, 1, 2$)
→ Find $\text{Var}[X]$.

Solution: $X = X_1 + X_2$, $X_1 = \begin{cases} 1 & \text{draw red first} \\ 0 & \text{draw yellow first} \end{cases}$, X_2 sym.

called "indicator" variables / "indicator method"

$$X_i \sim \text{Ber}\left(\frac{3}{5}\right) \quad (\text{more confuse for } i=2 \dots)$$

$$\text{Var}[X] = \text{Var}[X_1] + \text{Var}[X_2] + 2\text{Cov}[X_1, X_2]$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ p \cdot q & & p \cdot q \end{array} \quad \rightarrow p \quad \rightarrow p$$

$$\text{Cov}[X_1, X_2] = E[X_1 X_2] - E[X_1] E[X_2]$$

$$\text{Study: } X_1 X_2 = \begin{cases} 1 & \text{if } X_1=1, X_2=1 \\ 0 & \text{else} \end{cases}, \quad X_1, X_2 \sim \text{B}\left(\frac{3}{5}, \frac{3}{5}\right)$$

$$\text{So } E[X_1 X_2] = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$

$$\text{Var}[X] = \frac{3}{5} \left(1 - \frac{3}{5}\right) + \frac{3}{5} \left(1 - \frac{3}{5}\right) + 2 \left(\frac{3}{10} - \frac{3}{5} \cdot \frac{3}{5}\right) = \boxed{\frac{9}{25}}$$