Conditional Distributions

Review: Conditioning with **events**:

$$P'[-] = P[-|A] = \frac{P[-\cap A]}{P[A]}$$

$$P[AB] = P[B|A]P[A]$$

$$P[B] = P[B|A_1]P[A_1] + \dots + P[B|A_n]P[A_n]$$

partition: $\begin{cases} A_i \cap A_j = \emptyset \\ \bigcup A_i = S \end{cases}$

Extend to distributions:

$\boxed{\text{Fixed Event } A}$

$A \subset \mathbb{R}$ i.e. set of potential X values.

$$P_{X|A}(k) = \begin{cases} \frac{1}{P[A]} P_X(k) & k \in A \\ 0 & k \notin A \end{cases}$$

$$P[A] = \sum_{k \in A} P_X(k)$$

$$f_{X|A}(x) = \begin{cases} \frac{1}{P[A]} f_X(x) & x \in A \\ 0 & x \notin A \end{cases}$$

$$F_{X|A}(x) = P[X \le x | A], \quad f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

$$P_X(k) = P_{X|A_1}(k) \cdot P[A_1] + \dots + P_{X|A_n}(k) \cdot P[A_n]$$

$$f_X(x) = f_{X|A_1}(x) \cdot P[A_1] + \dots + f_{X|A_n}(x) \cdot P[A_n]$$

Variable Condition

Suppose X, Y RVs. The distribution of X conditioned on Y describes the probabilities of X given $Y = y$ (for each $y \in \mathbb{R}$).

$$p_{X|Y}(k|\ell) = P[X = k \mid Y = \ell]$$

$$= \frac{p_{X,Y}(k,\ell)}{p_Y(\ell)}$$

$p_{X|A}(k)$

$A = $ "$Y = y$"

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Notice:

$$p_{X,Y}(k,\ell) = p_{X|Y}(k|\ell) \cdot p_Y(\ell)$$

$$f_{X,Y}(k,\ell) = f_{X|Y}(k|\ell) \cdot f_Y(\ell)$$

Example: Say $P_{X,Y}(k,\ell) = \begin{cases} \frac{k+\ell}{21} & k=1,2,3;\ \ell=1,2 \\ 0 & \text{else} \end{cases}$

Find $P_{X|Y}(k|\ell)$ and $P_{Y|X}(\ell|k)$.

Solution:

Marginals: $$P_X(k) = \sum_{\ell=1}^{2} P_{X,Y}(k,\ell) = \frac{k+1}{21} + \frac{k+2}{21} = \frac{2k+3}{21}$$

$k = 1,2,3$

$$P_Y(\ell) = \frac{3\ell+6}{21} = \frac{\ell+2}{7}, \quad \ell = 1,2$$

So: $$P_{X|Y} = \frac{P_{X,Y}}{P_Y} \rightsquigarrow \frac{k+\ell}{3\ell+6} \qquad \begin{matrix} k = 1,2,3 \\ \ell = 1,2 \end{matrix}$$

$$P_{Y|X} = \frac{P_{X,Y}}{P_X} \rightsquigarrow \frac{k+\ell}{2k+3} \qquad \begin{matrix} k = 1,2,3 \\ \ell = 1,2 \end{matrix}$$

Conditional Expectation

Fixed event: $E[X \mid A] = \sum k \, P_{X|A}(k)$

$$-OR- \quad = \int_{-\infty}^{\infty} x \, f_{X|A}(x) \, dx$$

$$E[g(X) \mid A] = \sum g(k) \, P_{X|A}(k)$$

$$-OR- \quad = \int_{-\infty}^{\infty} g(x) \, f_{X|A}(x) \, dx$$

Variable event: $A = \text{"}Y = y\text{"}$

$$E[X \mid Y = y] = \sum k \, P_{X|Y}(x|y)$$

$$= \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$$

$$E[g(X) \mid Y = y] = \sum g(k) \, P_{X|Y}(x|y)$$

$$= \int_{-\infty}^{\infty} g(x) \, f_{X|Y}(x|y) \, dx$$

$$Var[X|A] = E[(X-\mu_{X|A})^2 | A]$$
$$= E[X^2|A] - E[X|A]^2$$

Cases / Total Prob:

$$E[X] = E[X|A_1]P[A_1] + \dots + E[X|A_n]P[A_n]$$

Example: Suppose X measure lengths of items.

$$P_X(k) = \begin{cases} 0.15 & k = 1,2,3,4 \\ 0.1 & k = 5,6,7,8 \\ 0 & \text{else} \end{cases}$$

Suppose L = "length is at least 5"

(a) Find $P_{X|L}$ (b) $E[X|L]$, $Var[X|L]$

Solution:

(a) $P_{X|L}(k) = \begin{cases} 1/4 & k = 5,6,7,8 \\ 0 & \text{else} \end{cases}$ because:

$$P_{X|L}(k) = \begin{cases} \frac{P_X(k)}{P[L]} & k \in L = \{5,6,7,8\} \\ 0 & k \notin L \end{cases}$$

$$P[L] = P_X(5) + P_X(6) + \dots + P_X(8) = .1+.1+.1+.1 = .4$$

$$\rightsquigarrow \frac{.1}{.4} = 1/4$$

(b) $E[X|L] = 6.5 = \frac{1}{4}\cdot 5 + \dots + \frac{1}{4}\cdot 8$

$E[X^2|L] = \frac{1}{4}25 + \frac{1}{4}\cdot 36 + \frac{1}{4}\cdot 49 + \frac{1}{4}\cdot 64 = 43.5$

So $Var[X|L] = 43.5 - (6.5)^2 = \boxed{1.25}$

Example:

$$f_{X,Y} = \begin{cases} \frac{1}{y} e^{-x/y} e^{-y} & x, y \in (0, \infty) \\ 0 & \text{else} \end{cases}$$

Find $E[X \mid Y = y]$.

Solution:

Want $f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}}{f_Y} & x, y \in (0, \infty) \\ 0 & \text{else} \end{cases}$

- Need $f_Y(y) = \int_0^\infty \frac{1}{y} e^{-x/y} e^{-y} \, dx \rightsquigarrow e^{-y}$
- Then $\frac{f_{X,Y}}{f_Y} = \frac{\frac{1}{y} e^{-x/y} e^{-y}}{e^{-y}} = \frac{1}{y} e^{-x/y}$
- $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \, dx \rightsquigarrow \int_0^\infty \frac{x}{y} e^{-x/y} \, dx \rightsquigarrow \boxed{y}$

$E[X|Y] = g(Y)$ where $g(y) = E[X|Y=y]$

Then $E[X|Y]$ is an RV encoding the expected X as spread over the possible Y.

"Iterated Expectation Theorem"

$$E[\,E[X|Y]\,] = E[X]$$

E.g. roll die $N = 1, 2, \ldots, 6$

flip coin N times, $X = \#$ heads.

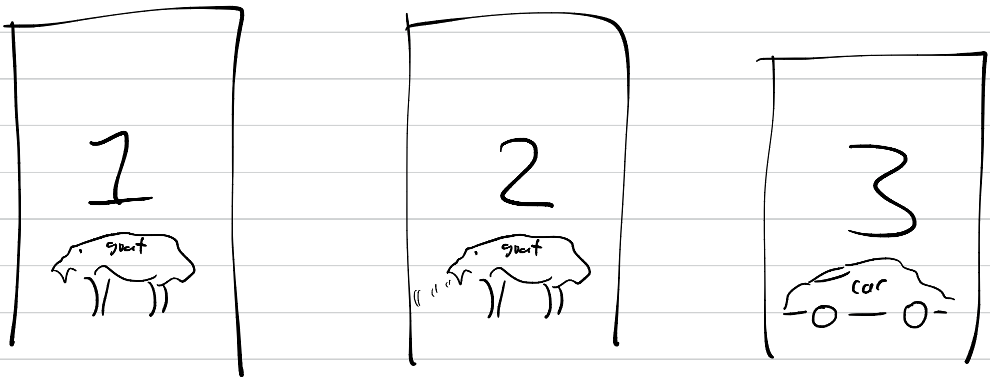
Then $E[X|N=n] = \begin{cases} 0.5 & n=1 \\ 1.0 & 2 \\ 1.5 & 3 \\ 2.0 & 4 \\ 2.5 & 5 \\ 3.0 & 6 \end{cases} = \frac{1}{2}n$

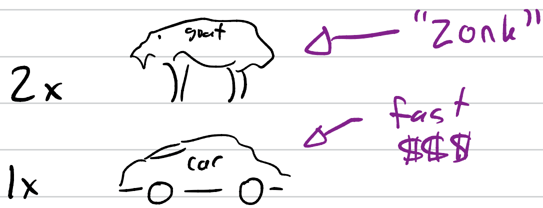
So $E[X|N] = \frac{1}{2}N$

So $E[X] = E[\,E[X|N]\,]$

$\rightsquigarrow E[\frac{1}{2}N] = \frac{1}{2}E[N] \rightsquigarrow \frac{1}{2}\cdot\frac{7}{2} = \boxed{\frac{7}{4}}$

Monty Hall Problem





Strat. 1 : stick : $1/3$ for car

Strat. 2 : switch : $2/3$ for initial goat $\Rightarrow$ final car