Summations

$$X = X_1 + \ldots + X_n = \sum_{i=1}^{n} X_i$$

$$E[X] = E[X_1] + \ldots + E[X_n] = \sum_{i=1}^{n} E[X_i]$$

(dependent or independent)

know $E[X_1 + X_2] = E[X_1] + E[X_2]$

$E[(X_1+X_2) + X_3] = E[X_1+X_2] + E[X_3]$ etc.

$$Var[X] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i<j} Cov[X_i, X_j]$$

know $Var[X_1+X_2] = Var[X_1] + Var[X_2] + 2\,Cov[X_1, X_2]$

like $(X_1+X_2)\cdot(X_1+X_2) = X_1\cdot X_1 + X_2\cdot X_2 + X_1\cdot X_2 + X_2\cdot X_1$

so extend:
$$\begin{aligned}(X_1+\ldots+X_n)\cdot(X_1+\ldots+X_n) &= X_1\cdot X_1 + X_1\cdot X_2 + X_1\cdot X_3 + \ldots \\ &+ X_2\cdot X_1 + X_2\cdot X_2 + X_2\cdot X_3 + \ldots \\ &+ X_3\cdot X_1 + X_3\cdot X_2 + X_3\cdot X_3 + \ldots \\ &\vdots\end{aligned}$$

Recall Cov bilinearity $\rightsquigarrow$ FOIL
$$\begin{aligned}&Cov[X_1+X_2, X_1+X_2] \\ &= Cov[X_1, X_1] + Cov[X_1, X_2] \\ &+ Cov[X_2, X_1] + Cov[X_2, X_2]\end{aligned}$$

Examples: (a) Binomial μ, σ (b) Pascal μ, σ

Solution: (a) $X_i \sim Ber(p)$, $X_i = \begin{cases} 1 & \text{success} \\ 0 & \text{fail} \end{cases}$

$$P_{X_i}(k) = \begin{cases} p & k=1 \\ q & k=0 \end{cases}$$

$X = \sum_{i=1}^{n} X_i$ = # success events $X \sim Bin(n, p)$

$E[X] = \sum_{i=1}^{n} E[X_i]$, $E[X_i] = 1 \cdot p + 0 \cdot q = p$

$$\rightsquigarrow \sum E[X_i] = \boxed{n \cdot p = \mu_X}$$

$Var[X] = \sum_{i=1}^{n} Var[X_i]$ because X_i, X_j indep. $i \neq j$.

$$Var[X_i] = E[\underset{\overset{\shortparallel}{X_i}}{X_i^2}] - E[X_i]^2 = p - p^2 = p(1-p) = pq$$

$$\rightsquigarrow \boxed{Var[X] = npq, \quad \sigma_X = \sqrt{npq}}$$

(b) $X = \sum_{i=1}^{\ell} X_i$, $X_i \sim Geo(p)$, $X \sim Pascal(\ell, p)$

$E[X_i] = \frac{1}{p}$, $Var[X_i] = \frac{q}{p^2}$

$$\rightsquigarrow \boxed{E[X] = \ell \cdot \frac{1}{p}, \quad \text{by indep. } Var[X] = \ell \frac{q}{p^2}}$$

Example: Form study groups of 10 students from a large population. Assume a "month" = 1/12 of year, all equal.

Q: How many months are expected to have a birthday from a group member in a random group?

Solution: Let $X = \sum_{i=1}^{12} X_i$ $\qquad X_i = \begin{cases} 1 & \text{if month } i \text{ has b-day} \\ 0 & \text{else} \end{cases}$

Want $E[X_1 + \dots + X_{12}] = 12 \cdot E[X_i]$ any i

Given i, $P[\geq 1 \text{ b-day in } i]$

$= 1 - P[\text{no b-day in } i]$

$= 1 - \left(\frac{11}{12}\right)^{10}$ $\qquad$ so $P_{X_i} = \begin{cases} 1 - \left(\frac{11}{12}\right)^{10} & k=1 \\ 0 & 0 \end{cases}$

$$E[X_i] = 1 \cdot P_{X_i}(1) + 0 \cdot P_{X_i}(0) = 1 - \left(\frac{11}{12}\right)^{10}$$

So $\boxed{\text{ANS} = 12\left(1 - \left(\frac{11}{12}\right)^{10}\right)}$

<u>Example:</u> "hats in the air"

Say n sailors throw their hats in air, catch random hat.

(a) How many sailors expected to catch own hat.

(b) Variance of that number?

<u>Solution:</u> (a) = (b) = 1 regardless of n.

(a) Let $X = \sum X_i$, $X_i = \begin{cases} 1 & \text{if sailor } i \text{ catches own hat} \\ 0 & \text{else} \end{cases}$

So X = # own hats caught

$X_i \sim Ber(p)$

$$P_{X_i}(k) = \begin{cases} 1/n & k=1 \\ \frac{n-1}{n} & k=0 \end{cases} \quad \text{so } E[X_i] = \frac{1}{n}$$

Thus $\boxed{E[X] = n \cdot \frac{1}{n} = 1}$

(b) $Var[X] = \sum Var[X_i] + 2\sum_{i<j} Cov[X_i, X_j]$

$Var[X_i] = E[X_i^2] - E[X_i]^2$. $\quad X_i^2 = X_i = \begin{cases} 1 & p = 1/n \\ 0 & n-1/n \end{cases}$

$\hookrightarrow \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$

$1/n$

$Cov[X_i, X_j] = E[X_i X_j] - E[X_i]E[X_j]$

Note $X_i X_j = \begin{cases} 1 & \begin{array}{l} i \text{ catches } i^{th} \\ j \text{ catches } j^{th} \end{array} \\ 0 & \text{else} \end{cases}$

$$P[X_i X_j = 1] = \frac{1}{n} \cdot \frac{1}{n-1}$$

So $E[X_i X_j] = \frac{1}{n(n-1)}$

$$\frac{1}{n(n-1)} - \frac{1}{n^2} = \frac{n-(n-1)}{n^2(n-1)}$$

$$\rightsquigarrow Var[X] = \sum_{i=1}^{n} \frac{n-1}{n^2} + 2 \sum_{i<j} \left(\frac{1}{n(n-1)} - \frac{1}{n} \cdot \frac{1}{n} \right)$$

$$\rightsquigarrow \frac{n-1}{n} + 2 \left(\sum_{i<j} \frac{1}{n^2(n-1)} \right)$$

There are $\frac{n(n-1)}{2}$ pairs $i<j$.

$$\rightsquigarrow \frac{n-1}{n} + 2 \frac{n(n-1)}{2} \cdot \frac{1}{n^2(n-1)} \rightsquigarrow \boxed{1}$$