"Statistical Inference Theory"

Significance Testing

"unary" hypothesis testing

Significance Test

- Null hypothesis H_0 (background assumption)
  - Identify a Claim ⟿ H_0 less likely

    wish to invalidate H_0 in favor of Claim

- Rejection Region R ("decision rule")
  - Need a "decision statistic" X, an RV
  - Write R in terms of X values
  - "Significance level" α
  - H_0 ⟿ R less likely, Claim ⟿ R more likely

- $P[R \mid H_0]$
  - compute somehow
  - usually infer from $f_{X|H_0}$, $P_{X|H_0}$
  - adjust R so that $P[R \mid H_0] = \alpha$.

$$\alpha = P[R \mid H_0]$$
$$= P[\text{reject } H_0 \mid H_0 \text{ is true}] = \text{"Type I error"}$$

	H_0 is true	H_0 is false
Maintain H_0	Good call	Type II error
Reject H_0	Type I error	Good call

"H_0 innocent until proven guilty"

Example: Your friend gives you a die, worries that it's weighted to come up 2 more often. Design a sign. test on data of 20 rolls to decide whether weighted, $\alpha = 5\%$.

Solution:

- H_0 = die is fair
  - Claim: 2 is more likely/frequent than $1/6$

- $R = ?$ (need to compute)
  - $X = \#$ 2's. $X|H_0 \sim \mathrm{Bin}(20, 1/6)$
  - $R = \{x \mid x \geq r\}$ some r (need r)

- $P[R|H_0] = P[X \geq r \mid H_0] = \alpha = .05$

Use $P_{X|H_0}(k) = \binom{20}{k} (1/6)^k (5/6)^{20-k}$
= prob. of k twos

Now, rewrite: $P[X < r \mid H_0] = 0.95$

k	0	1	2	3	4	5	6	7	...
$F_{X\|H_0}$	.026	.130	.329	.567	.769	.898	.963	.989	

$\Rightarrow$ choose $r = 6$

$\rightsquigarrow P[X \geq 6 \mid H_0] < .05$

but $P[X \geq 5 \mid H_0] \nless .05$

$$\boxed{R = \{X \mid X \geq 6\}}$$

Example: AC circuit nominal 130 V, $\sigma = 2.1$. Design a two-tail sign. test based on 40 measurements to determine whether the true avg. voltage is 130, $\alpha = 2\%$.

Solution:

- $H_0 =$ "$E[V] = 130$"

  Claim $E[V] \neq 130$

- $X = M_{40}(V) \qquad V \sim ??$

- $R = |X - 130| \geq c$ find c s.t.

$$P[\,|X-130| \geq c\,] = 0.02$$

Use Chebyshev:

$$P[\,|X-130| \geq c\,] \leq \frac{\sigma_X^2}{c^2} = \frac{\sigma_V^2}{40 \cdot c^2}$$

$$\sigma_V = 2.1$$

Solve $\dfrac{(2.1)^2}{40 \cdot c^2} = 0.02 \leadsto c = 2.348$

$$\Rightarrow \boxed{R = \{X \mid X < 127.65 \text{ OR } X > 132.35\}}$$

Binary Hypothesis Testing

Ingredients of test:

- H_0 & $H_1 = H_0^c$
  - Maybe know $P[H_0]$, $P[H_1]$ (MAP, MC)
- A_0 & $A_1 = A_0^c$
  - $H_0 \rightsquigarrow A_0$ likely, $H_1 \rightsquigarrow A_1$ likely
  - Test rule: decide H_0 or H_1
  - Write A_0 using decision statistic X
- Choose a design type: MAP, ML, MC

MAP Design "maximum a posteriori probability"

Suppose we know $P[H_0]$, $P[H_1]$
and $P_{X|H_0}(x)$, $P_{X|H_1}(x)$ (or $f_{X|H_0}$ ...)

MAP design: A_0 = set of all x such that

discrete: $P_{X|H_0}(x) \cdot P[H_0] \geq P_{X|H_1}(x) P[H_1]$

cts: $f_{X|H_0}(x) P[H_0] \geq f_{X|H_1}(x) P[H_1]$

$$A_1 = \{x \in \mathbb{R} \mid x \notin A_0\}$$

This minimizes the total probability of error.

ML Design

(Don't know priors $P[H_0]$, $P[H_1]$.)

ML design: A_0 = all x such that

discrete: $P_{X|H_0}(x) \geq P_{X|H_1}(x)$

cts: $f_{X|H_0}(x) \geq f_{X|H_1}(x)$

Like: simplify MAP by either

view as:
- set $P[H_0] = P[H_1]$ and cancel
- ignore priors

Can use as frequentist!

Type I error, false alarm rate:

$$P_{FA} = P[A_1 \mid H_0]$$

Type II error, miss rate:

$$P_{Miss} = P[A_0 \mid H_1]$$

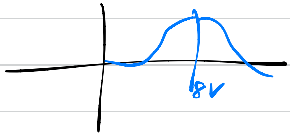
Total error rate: $P_{FA} \cdot P[H_0] + P_{Miss} \cdot P[H_1]$

Example: ML Test, smoke detectors

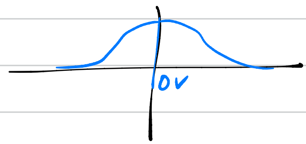
Suppose sensor makes 8V when no smoke, 0V when smoke

But... background noise: $\mathcal{N}(0, 3^2)$

On:



Off:



- Design ML test to sound alarm.
- What are the three error rates?

Solution:

$$X|H_0 \sim \mathcal{N}(0, 3^2)$$

$$X|H_1 \sim \mathcal{N}(8, 3^2)$$

$$f_{X|H_0} = \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2}, \quad f_{X|H_1} = \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}$$

ML condition:

$$\frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \gtrless \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}$$

$$\rightarrow \quad x \leq 4 = A_0, \quad x > 4 = A_1$$

Type I error:

$$P_{FA} = P[A_1 | H_0] \rightsquigarrow P[X > 4 | H_0]$$

$$\rightsquigarrow 1 - P\left[\frac{X-0}{3} \le \frac{4}{3} \,\middle|\, H_0\right]$$

$$\rightsquigarrow 1 - P[Z \le 1.33]$$

$$\rightsquigarrow \boxed{0.0912}$$

Type II error:

$$P_{Miss} = P[A_0 | H_1] \rightsquigarrow P[X \le 4 | H_1]$$

$$\rightsquigarrow P\left[\frac{X-8}{3} \le \frac{4-8}{3} \,\middle|\, H_1\right]$$

$$\rightsquigarrow P[Z \le -1.33]$$

$$\rightsquigarrow \boxed{0.0912}$$

$$P_{ERR} = P_{FA} \cdot 0.5 + P_{Miss} \cdot 0.5$$

$$\rightsquigarrow \boxed{0.0912}$$