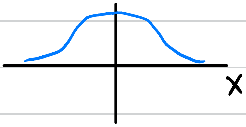
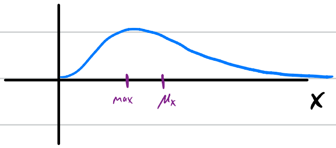
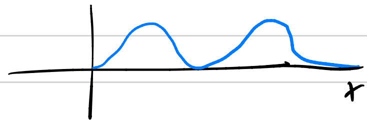
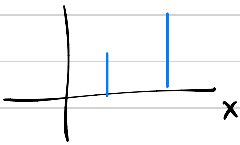
Mean Square Error

Say we have a distribution:









Mean Square Error of guess $\hat{x}$ is:

cost function: $E[(X-\hat{x})^2]$ = expected (square) error of my guess $\hat{x}$

Turns out... $\hat{x} = E[X]$ minimizes MSE.
@ $\hat{x} = E[X] \rightsquigarrow MSE = Var[X]$.

Why? $E[(X-\hat{x})^2] \rightsquigarrow E[X^2 - 2\hat{x}X + \hat{x}^2]$

$\rightsquigarrow E[X^2] - 2E[X]\hat{x} + \hat{x}^2$

minimize: $\frac{d}{d\hat{x}}\left(\text{" "} \right) = -2E[X] + 2\hat{x}$ set $=0$

$\rightsquigarrow \boxed{\hat{x} = E[X]}$

parabola => global min here.

$\hat{x}_B = E[X]$ = "blind estimate of X no other info"
$e_B = Var[X]$ = expected error of $\hat{x}_B$.

$\hat{x}_A = E[X|A]$ = MSE estimate given A
$e_{X|A} = Var[X|A]$

$\hat{x}_M(y) = E[X|Y=y]$

Example:

1	2	3	4	5
.15	.28	.26	.19	.13

Say X is even. Find min. MSE estimate & error.

Solution: $A = \{2, 4\}$

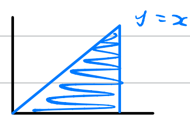
ANS $= E[X \mid A]$

$$P_{X|A}(k) = \begin{cases} .19/.47 & k=4 \\ .28/.47 & 2 \\ 0 & \text{else} \end{cases}$$

So $\hat{x}_A = 2\frac{.28}{.47} + 4\frac{.19}{.47} \rightsquigarrow 2.81$

$e_{X|A} = (2-2.81)^2\frac{.28}{.47} + (4-2.81)^2\frac{.19}{.47} \rightsquigarrow .96$

Example: $f_{X,Y} = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{else} \end{cases}$



Find min. MSE of X in terms of Y.
Estimate X when $Y = .2, .8$.

Solution: $\hat{x}_M(y) = E[X \mid Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y=y}(x|y)\, dx$

$f_{X|Y=y} = \frac{f_{X,Y}}{f_Y}$, $f_Y = \int_{x=y}^{1} 8xy\, dx = 4y(1-y^2)$

$f_{X|Y=y} = \frac{8xy}{4y(1-y^2)} = \frac{2x}{1-y^2}$ $\quad 0 \le y \le x \le 1$

$\hat{x}_M = \int_{x=y}^{1} x \frac{2x}{1-y^2}\, dx \rightsquigarrow \boxed{\frac{2}{3}\frac{1-y^3}{1-y^2}}$

plug in $y = .2$ or $y = .8$

Line of min. MSE

AKA "Linear least squares estimator"

Have X, Y; want $\hat{X}_L(Y) = aY + b$

$$L(y) = ay + b$$

error: $e_L(a,b) = E\left[\left(X - \hat{X}_L(Y)\right)^2\right]$

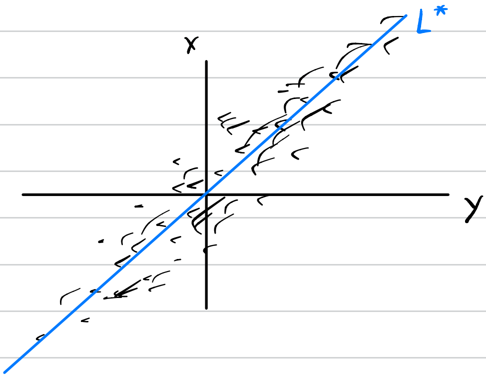
"measured" − "predicted X"

minimize $e_L(a,b)$ value over (a,b) pairs

$$\rightsquigarrow a^* = \rho_{X,Y} \frac{\sigma_X}{\sigma_Y} = \frac{Cov[X,Y]}{Var[Y]}, \quad b^* = \mu_X - a^* \mu_Y$$

$$@ \ (a^*, b^*), \quad e_{L^*} = \sigma_X^2 (1 - \rho_{X,Y}^2)$$

FACT "$X - \hat{X}_{L^*}(Y)$" as RV, is uncorrelated with Y.



Example: Suppose $R \sim \text{Unif}(0,1)$, $X \sim \text{Unif}(0,R)$

(a) $\hat{x}_M(r)$ (b) $\hat{r}_M(x)$ (c) $\hat{R}_{L\min}(X)$

Solution:

(a) $\hat{x}_M(r) = E[X \mid R = r]$

Given $R = r$, $X \sim \text{Unif}(0, r)$

so $E[X \mid R = r] = \boxed{\frac{r}{2}}$

(b) $\hat{r}_M(x) = E[R \mid X = x]$

This time need $f_{R|X}(r|x) = \frac{f_{X,R}}{f_X}$

Have $f_R = 1$ for $0 < r < 1$, $= 0$ else

$$f_{X|R}(x|r) = \begin{cases} 1/r & 0 < x < r \\ 0 & \text{else} \end{cases}$$

$$\rightsquigarrow f_{X,R} = f_{X|R} \cdot f_R = \begin{cases} 1/r & 0 < x < r < 1 \\ 0 & \text{else} \end{cases}$$

$$\rightsquigarrow f_X = \int_x^1 \frac{1}{r}\, dr \rightsquigarrow -\ln x \qquad (0 < x < 1)$$

$$\rightsquigarrow f_{R|X}(r|x) = \begin{cases} \frac{-1}{r \ln x} & 0 < x < r < 1 \\ 0 & \text{else} \end{cases}$$

$$\rightsquigarrow E[R \mid X = x] = \int_x^1 r \left(\frac{-1}{r \ln x}\right) dr \rightsquigarrow \boxed{\frac{x-1}{\ln x} = \hat{r}_M(x)}$$

(c) $\hat{R}_{Lmm}(X)$ Get all stats...

$E[R] = \frac{1}{2}$ $R \sim Unif(0,1)$

$\sigma_R^2 = \frac{(b-a)^2}{12} = \frac{1}{12}$

$E[X] = \int_0^1 x(-\ln x)\,dx \rightsquigarrow \frac{1}{4}$

$\sigma_X^2 = \int_0^1 x^2(-\ln x)\,dx = \frac{7}{144}$

$E[XR] = \int_{x=0}^{1}\int_{r=x}^{1}(xr)\left(\frac{1}{r}\right)dr\,dx \rightsquigarrow \frac{1}{6}$

$= \int_{r=0}^{1}\int_{x=0}^{r}(xr)\frac{1}{r}\,dx\,dr$

So $Cov[X,R] = E[XR] - E[X]E[R] \rightsquigarrow \frac{1}{24}$

$a^* = \rho_{X,Y}\frac{\sigma_X}{\sigma_Y} = \frac{Cov[X,Y]}{Var[Y]}$, $b^* = \mu_X - a^*\mu_Y$

So $a^* = \frac{Cov[X,R]}{\sigma_X^2} = \frac{1/24}{7/144} \rightsquigarrow \frac{6}{7}$

$b^* = E[R] - \frac{6}{7}E[X] \rightsquigarrow \frac{1}{2} - \frac{6}{7}\left(\frac{1}{4}\right) \rightsquigarrow \frac{2}{7}$

$$\boxed{\hat{R}_{Lmm}(X) = \frac{6}{7}X + \frac{2}{7}}$$